Undergraduate problems

U259. Compute

$$\lim_{n\to\infty} \frac{\left(1+\frac{1}{n(n+a)}\right)^{n^3}}{\left(1+\frac{1}{n(n+b)}\right)^{n^3}}.$$

Proposed by Arkady Alt, San Jose, California, USA

Solution by Giulio Calimici, and Emiliano Torti, Università di Roma "Tor Vergata", Roma, Italy

We note that

$$n^3 \ln\left(1+\frac{1}{n(n+a)}\right) = n^3\left(\frac{1}{n(n+a)} + O(1/n^4)\right) = n\left(\frac{1}{1+a/n} + O(1/n^2)\right)$$
$$= n\left(1-\frac{a}{n} + O(1/n^2)\right) = n-a+O(1/n).$$

Hence

$$\frac{\left(1+\frac{1}{n(n+a)}\right)^{n^3}}{\left(1+\frac{1}{n(n+b)}\right)^{n^3}} = \exp\left(n^3\ln\left(1+\frac{1}{n(n+a)}\right) - n^3\ln\left(1+\frac{1}{n(n+b)}\right)\right)$$
$$= \exp\left(n-a+O(1/n)-n+b+O(1/n)\right) \to \exp(b-a) = e^{b-a}.$$

Also solved by Albert Stadler, Switzerland; Stanescu Florin, Serban Cioculescu school, Gaesti, Romania; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Moubinool Omarjee Lycée Henri IV , Paris , France; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Alessandro Ventullo, Milan, Italy; Anastasios Kotronis, Athens, Greece; Robin Park, Thomas Jefferson High School For Science and Technology; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Roma, Italy; Radouan Boukharfane, Polytechnique de Montreal, Canada; Li Zhou, Polk State College, USA.