Undergraduate problems

U259. Compute

$$\lim_{n\to\infty} \frac{\left(1+\frac{1}{n(n+a)}\right)^{n^3}}{\left(1+\frac{1}{n+b}\right)^{n^2}}.$$

Proposed by Arkady Alt , San Jose, California, USA

U260. Let $A, B \in M_n(\mathbb{R})$ be symmetric positive definite matrices. Prove that
(a) $\operatorname{tr}(A^2 + AB^2A) = \operatorname{tr}(A^2 + BA^2B)$.
(b) $\operatorname{tr}[(A^2 + AB^2A)^{-1}] \geq \operatorname{tr}[(A^2 + BA^2B)^{-1}]$.

Proposed by Cosmin Pohoata, Princeton University, USA

U261. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ which are derivatives and satisfy

$$f^2 \in \int f(x)dx.$$

Proposed by Mihai Piticari, "Dragos Voda" National College, Romania

U262. Let $T_n(x)$ be the sequence of Chebyshev polynomials of the first kind, defined by $T_0(x) = 0$, $T_1(x) = x$, and

$$T_{n+1} = 2xT_n(x) - T_{n-1}(x)$$

for $n \geq 1$. Prove that for all $x \geq 1$ and all positive integers n

$$x \leq \sqrt[n]{T_n(x)} \leq 1 + n(x-1).$$

Proposed by Arkady Alt , San Jose, California, USA

U263. Let $n \geq 2$ be an integer. A general $n \times n$ magic square is a matrix $A \in M_n(\mathbf{R})$ such that the sum of the elements in each row of A is the same. Prove that the set of $n \times n$ general magic squares is an $\mathbf{R}$-vector space and find its dimension.

Proposed by Cosmin Pohoata, Princeton University, USA

U264. Let A be a finite ring such that $1 + 1 = 0$. Prove that the equations $x^2 = 0$ and $x^2 = 1$ have the same number of solutions in A.

Proposed by Mihai Piticari, "Dragos Voda" National College, Romania