U190. Evaluate

$$\lim_{n\to\infty}\left(n\frac{\sqrt[n+1]{(n+1)!)}}{\sqrt[n]{n!}}-(n-1)\frac{\sqrt[n]{n!)}}{\sqrt[n-1]{(n-1)!}}\right).$$

Proposed by Arkady Alt, San Jose, California, USA

Solution by the author

We will use inequality $\left(\frac{n}{e}\right)^n\sqrt{an}<n!<\left(\frac{n}{e}\right)^n\sqrt{an}\cdot e^{\frac{1}{12n}}$.

Let $L_n:=\frac{n}{e}\sqrt[2n]{an}, R_n:=L_n\cdot e^{\frac{1}{12n^2}}, a_n=n\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}-(n-1)\frac{\sqrt[n]{n!}}{\sqrt[n-1]{(n-1)!}}$.

Since $L_n<\sqrt[n]{n!}<R_n$ then $n\frac{L_{n+1}}{R_n}-(n-1)\frac{R_n}{L_{n-1}}<a_n<n\frac{R_{n+1}}{L_n}-(n-1)\frac{L_n}{R_{n-1}}$.

1. We will prove that $\lim_{n\to\infty} n\left(\frac{L_{n+1}}{R_n}-\frac{L_{n+1}}{L_n}\right)=0$.

Noting that $\lim_{n\to\infty}\frac{L_{n+1}}{L_n}=\lim_{n\to\infty}\frac{(n+1)\sqrt[2n+2]{a(n+1)}}{n\sqrt[2n]{an}}=1$ we obtain

$\lim_{n\to\infty} n\left(\frac{L_{n+1}}{R_n}-\frac{L_{n+1}}{L_n}\right)=\lim_{n\to\infty}\frac{nL_{n+1}}{R_n}\left(1-e^{\frac{1}{12n^2}}\right)=$

$\lim_{n\to\infty}\frac{nL_{n+1}}{12n^2L_n}\cdot\lim_{n\to\infty}12n^2\left(1-e^{\frac{1}{12n^2}}\right)\cdot\lim_{n\to\infty}e^{-\frac{1}{12n^2}}=-\frac{1}{12}\lim_{n\to\infty}\frac{1}{n}\cdot\frac{L_{n+1}}{L_n}=0.$

Similarly we obtain $\lim_{n\to\infty} n\left(\frac{R_{n+1}}{L_n}-\frac{L_{n+1}}{L_n}\right)=0$

2. Using **1.** we obtain $\lim_{n\to\infty}\left(n\frac{L_{n+1}}{R_n}-(n-1)\frac{R_n}{L_{n-1}}\right)=$

$\lim_{n\to\infty}\left(n\frac{L_{n+1}}{L_n}-(n-1)\frac{L_n}{L_{n-1}}+n\left(\frac{L_{n+1}}{R_n}-\frac{L_{n+1}}{L_n}\right)-(n-1)\left(\frac{R_n}{L_{n-1}}-\frac{L_n}{L_{n-1}}\right)\right)=$

$\lim_{n\to\infty}\left(n\frac{L_{n+1}}{L_n}-(n-1)\frac{L_n}{L_{n-1}}\right)$ and $\lim_{n\to\infty}\left(n\frac{R_{n+1}}{L_n}-(n-1)\frac{L_n}{R_{n-1}}\right)=$

$\lim_{n\to\infty}\left(n\frac{L_{n+1}}{L_n}-(n-1)\frac{L_n}{L_{n-1}}\right)$.Let $\alpha_n:=\ln\frac{L_{n+1}}{L_n}=\ln\left(\frac{(n+1)\sqrt[2n+2]{a(n+1)}}{n\sqrt[2n]{an}}\right)=$

$\ln\frac{(n+1)\sqrt[2n+2]{a(n+1)}}{n\sqrt[2n]{an}}=\ln\left(1+\frac{1}{n}\right)-\frac{\ln a}{n(n+1)}+\frac{\ln(n+1)}{2(n+1)}-\frac{\ln n}{2n}$ then

$n\frac{L_{n+1}}{L_n}=ne^{\alpha_n}$, $(n-1)\frac{L_n}{L_{n-1}}=(n-1)e^{\alpha_{n-1}}$ and we obtain

$\lim_{n\to\infty}\left(n\frac{L_{n+1}}{L_n}-(n-1)\frac{L_n}{L_{n-1}}\right)=\lim_{n\to\infty}\left(ne^{\alpha_n}-(n-1)e^{\alpha_n-1}\right)=$

$\lim_{n\to\infty}\left(n\left(e^{\alpha_n}-1\right)-(n-1)\left(e^{\alpha_{n-1}}-1\right)\right)+1.$

Note that $\lim_{n\to\infty} n\alpha_n=\lim_{n\to\infty} n\left(\ln\left(1+\frac{1}{n}\right)-\frac{\ln a}{n(n+1)}+\frac{\ln(n+1)}{2(n+1)}-\frac{\ln n}{2n}\right)=$

$=\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right)-\lim_{n\to\infty}\frac{\ln a}{n+1}+\frac{1}{2}\lim_{n\to\infty}\left(\frac{n\ln(n+1)}{(n+1)}-\ln n\right)=$

$$1+\frac{1}{2}\lim_{n\to\infty}\left(\frac{n\ln(n+1)}{(n+1)}-\ln n\right)=1+\frac{1}{2}\lim_{n\to\infty}\left(\frac{\ln\left(1+\frac{1}{n}\right)^{n}-\ln n}{n+1}\right)=1$$

Since $\lim\limits_{n\to\infty}\dfrac{e^{\alpha_n}-1}{\alpha_n}=1$ then $\lim\limits_{n\to\infty}n\left(e^{\alpha_n}-1\right)=\lim\limits_{n\to\infty}n\alpha_n\lim\limits_{n\to\infty}\dfrac{e^{\alpha_n}-1}{\alpha_n}=1$ and, therefore, $\lim\limits_{n\to\infty}\left(n\left(e^{\alpha_n}-1\right)-(n-1)\left(e^{\alpha_{n-1}}-1\right)\right)=1-1=0$.

Thus, $\lim\limits_{n\to\infty}a_n=1$.

Also solved by Omran Kouba, Institute for Applied Sciences and Technology, Syria; Lorenzo Pascali, Università di Roma "La Sapienza", Roma, Italy; Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Moubinool Omarhee, Paris, France; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy.