Undergraduate problems

U187. Let p be a a prime such that $p \equiv 3 (\text{mod } 8)$ or $p \equiv 5 (\text{mod } 8)$, and $p = 2q + 1$ with qa prime. Evaluate $\omega^2 + \omega^4 + \cdots + \omega^{2^{p-1}}$, where $\omega \neq 1$ is a root of order p of unity.

Proposed by Dorin Andrica and Mihai Piticari, Romania

U188. Let G be a finite group in which for every positive integer m the number of solutions in G of the equation $x^m = e$ is at most m. Prove that G is cyclic.

Proposed by Roberto Bosch Cabrera, Florida, USA

U189. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be distinct complex numbers such that $a_k + b_l \neq 0$ for all $k, l = 1, 2, \ldots, n$. Solve the system of equations

$$\frac{x_1}{a_k + b_1} + \frac{x_2}{a_k + b_2} + \cdots + \frac{x_n}{a_k + b_n} = \frac{1}{a_k}, \ k = 1, 2, \ldots, n.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

U190. Evaluate

$$\lim_{n \to \infty} \left(n \frac{\sqrt[n+1]{(n+1)!)}}{\sqrt[n]{n!}} - (n-1) \frac{\sqrt[n]{n!)}}{\sqrt[n-1]{(n-1)!}} \right).$$

Proposed by Arkady Alt, San Jose, California, USA

U191. For a positive integer n define $a_n = \prod_{k=1}^{n} \left(1 + \frac{1}{2^k}\right)$. Prove that

$$2 - \frac{1}{2^n} \leq a_n < 3 - \frac{1}{2^{n-1}}.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

U192. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with finite lateral limits at any point in $\mathbb{R}$. Prove that

(a) f is integrable on any interval $[a, b]$;

(b) If $F(x) = \int_0^x f(t)dt$ is differentiable at any point in $\mathbb{R}$, then f has finite limit at any point in $\mathbb{R}$.

Proposed by Sorin Radulescu and Mihai Piticari, Romania