U182. Find all continuous functions f on $[0,1]$ such that $f(x) = c$ if $x \in \left[0, \frac{1}{2}\right]$ and $f(x) = f(2x-1)$ if $x \in \left(\frac{1}{2}, 1\right]$, where c is a given constant.

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We will show that for every positive integer n

$$f(x) = c, x \in \left[0, 1 - \frac{1}{2^n}\right].$$

We will induct on n. For $n = 1$ it is the condition of the problem and therefore is true. Now suppose it is true for n. Let us prove it for $n+1$. If $x \in [\frac{1}{2}, 1 - \frac{1}{2^{n+1}}]$, then $2x - 1 \in [0, 1 - \frac{1}{2^n}] \Rightarrow$ (by induction hypothesis) $f(2x-1) = c \Rightarrow f(x) = c$. Thus $f(x) = c$ if $x \in [0, 1)$. Because f is continuous we have

$$f(1) = f\left(\lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right)\right) = \lim_{n\to\infty} f\left(\left(1 - \frac{1}{2^n}\right)\right) = \lim_{n\to\infty} c = c.$$

Then $f(x) = c$ for $x \in [0, 1]$.

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