Senior problems

S97. Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^n \geq \left(\sqrt[n]{x_1 x_2 \cdots x_n}\right)^{n-1} \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}.$$

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First solution by Manh Dung Nguyen, Hanoi University of Science, Vietnam

Without loss of generality we may assume that $x_1 + x_2 + \cdots + x_n = n$.
The inequality is equivalent to

$$(x_1 x_2 \cdots x_n)^{\frac{2(n-1)}{n}} \left(x_1^2 + x_2^2 + \cdots + x_n^2\right) \leq n$$

For n = 2, the inequality reduces to $(x_1 - x_2)^4 \geq 0$.
For $n \geq 3$, assume that $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ and apply the **EV- Theorem:**
For $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n, x_1 + x_2 + \cdots + x_n = n$ and $x_1^2 + x_2^2 + \cdots + x_n^2 =$ constant
The product $x_1 x_2 \cdots x_n$ is maximal when $0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n$
Consequently, it suffices to show the inequality for $x_1 = x_2 = \cdots = x_{n-1} = x$ and $x_n = y$ where $0 \leq x \leq 1 \leq y$ and $(n-1)x + y = n$. Under the circumstance, the inequality reduces to

$$x^{\frac{2(n-1)^2}{n}} y^{\frac{2(n-1)}{n}} \left[(n-1)x^2 + y^2\right] \leq n.$$

For $x = 0$, the inequality is trivial. For $x > 0$, it is equivalent to $f(x) \leq 0$ where

$$f(x) = \frac{2(n-1)^2}{n} \ln x + \frac{2(n-1)}{n} \ln y + \ln\left[(n-1)x^2 + y^2\right] - \ln n.$$

With $y = n - (n-1)x$. We have $y' = -(n-1)$ and

$$\frac{nf'(x)}{2(n-1)^2} = \frac{1}{x} - \frac{1}{y} + \frac{n(x-y)}{(n-1)\left[(n-1)x^2 + y^2\right]}$$
$$= \frac{(y-x)\left[\left[(n-1)x - y\right]^2 + (n-2)y(x+y)\right]}{(n-1)xy\left[(n-1)x^2 + y^2\right]} \geq 0.$$

Therefore, the function $f(x)$ is strictly increasing on $(0, 1]$ and hence $f(x) \leq f(1) = 0$. Equality occurs if and only if $x_1 = x_2 = \cdots = x_n$

Second solution by Michel Bataille, France

By homogeneity, we may suppose $x_1 x_2 \cdots x_n = 1$ and then prove

$$(x_1 + x_2 + \cdots + x_n)^{2n} \geq n^{2n-1} S$$

where $S = x_1^2 + x_2^2 + \cdots + x_n^2$. Using AM-GM,

$$\begin{aligned}(x_1 + x_2 + \cdots + x_n)^2 &= S + 2 \sum_{1 \leq i < j \leq n} x_i x_j \\ &\geq S + 2 \cdot \frac{n(n-1)}{2} \left((x_1 x_2 \cdots x_n)^{n-1}\right)^{2/(n(n-1))} \\ &= S + n(n-1),\end{aligned}$$

hence it suffices to show that

$$(S + n(n-1))^n \geq n^{2n-1} S.$$

Now, by AM-GM again,

$$S + n(n-1) = S + n + n + \cdots + n \geq n \left(S n^{n-1}\right)^{1/n}$$

and so

$$(S + n(n-1))^n \geq n^n S n^{n-1} = n^{2n-1} S,$$

as desired.

Also solved by Oles Dobosevych, Ukraine; Daniel Lasaosa, Universidad Publica de Navarra, Spain.