Senior problems

S97. Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^n \geq (\sqrt[n]{x_1x_2\cdots x_n})^{n-1}\sqrt{\frac{x_1^2+x_2^2+\cdots+x_n^2}{n}}.$$

Proposed by Arkady Alt, San Jose, California, USA

S98. Let n be a positve integer. Prove that $\prod_{d|n}\frac{\phi(d)}{d} \geq \left(\frac{\phi(n)}{n}\right)^{\frac{\tau(n)}{2}}$, where $\tau(n)$ is the number of divisors of n and $\phi(n)$ is Euler's totient function.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

S99. Let ABC be an acute triangle. Prove that

$$\frac{1-\cos A}{1+\cos A}+\frac{1-\cos B}{1+\cos B}+\frac{1-\cos C}{1+\cos C} \leq \left(1-\frac{1}{\cos A}\right)\left(1-\frac{1}{\cos B}\right)\left(1-\frac{1}{\cos C}\right).$$

Proposed by Daniel Campos Salas, Costa Rica

S100. Let ABC be an acute triangle with altitudes BE and CF. Points Q and R lie on segments CE and BF, respectively, such that $\frac{CQ}{QE}=\frac{FR}{RB}$. Determine the locus of the circumcenter of triangle AQR when Q and R vary.

Proposed by Alex Anderson, Washington University in St. Louis, USA

S101. Let a, b, c be distinct real numbers. Prove that

$$\left(\frac{a}{a-b}+1\right)^2+\left(\frac{b}{b-c}+1\right)^2+\left(\frac{c}{c-a}+1\right)^2 \geq 5.$$

Proposed by Roberto Bosch Cabrera, University of Havana, Cuba

S102. Consider triangle ABC with circumcenter O and incenter I. Let E and F be the points of tangency of the incircle with AC and AB, respectively. Prove that EF, BC, OI are concurrent if and only if $r_a^2=r_br_c$, where r_a, r_b, r_c are the radii of the excircles.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA