S262. Let a, b, c be the sides of a triangle and let m_a, m_b, m_c be the lengths of its medians. Prove that

$$a^2 + b^2 + c^2 - ab - bc - ca \leq 4\left({m_a}^2 + {m_b}^2 + {m_c}^2 - m_a m_b - m_b m_c - m_c m_a\right).$$

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We know that for a triangle ABC, if m_a,m_b,m_c are its medians on BC,CA,AB, so we have :

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}\left(a^2 + b^2 + c^2\right)$$

Taking account this identity, we are left to prove that :

$$\sum m_b m_c \leq \frac{1}{2}\sum a^2 + \frac{1}{4}\sum bc$$

We will prove that :

$$m_b m_c \leq \frac{a^2}{2} + \frac{bc}{b}$$

By squaring both sides of this inequality we get :

$$16 m_a^2 m_b^2 \leq (2a^2 + bc)^2$$

which is equivalent to :

$$16\frac{2(c^2+a^2)-b^2}{4}\cdot\frac{2(a^2+b^2)-c^2}{4} \leq (2a^2+bc)^2 \Leftrightarrow (b-c)^2(a+b+c)(a-b-c) \leq 0$$

Which is true. We used here the identities:

$$m_b^2 = \frac{2(c^2+a^2)-b^2}{4}$$

and

$$m_c^2 = \frac{2(a^2+b^2)-c^2}{4}$$

We prove similary:

$$m_a m_c \leq \frac{b^2}{2} + \frac{ac}{b}$$

and

$$m_a m_b \leq \frac{c^2}{2} + \frac{ab}{b}$$

By summing up the last inequalities we proved we get our result.

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