Senior problems

S259. Let a, b, c, d, e be integers such that

$$a(b+c)+b(c+d)+c(d+e)+d(e+a)+e(a+b)=0.$$

Prove that $a+b+c+d+e$ divides $a^5+b^5+c^5+d^5+e^5-5abcde$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S260. Let $m<n$ be positive integers and let $x_1, x_2, ..., x_n$ be positive real numbers. If A is a subset of $\{1,2,...,n\}$, define $s_A=\sum_{i\in A} x_i$ and $A^c=\{i\in\{1,2,\ldots,n\} | i\notin A\}$. Prove that

$$\sum_{|A|=m}\frac{s_A}{s_{A^c}}\geq\frac{m}{n-m}\binom{n}{m},$$

where the sum is taken over all m-element subsets A of $\{1,2,\ldots,n\}$.

Proposed by Mircea Becheanu, University of Bucharest, Romania

S261. Let ABC be a triangle with circumcircle Γ and let $\mathcal{K}$ be the circle simultaneously tangent to AB, AC and Γ, internally. Let X be a point on the circumcircle of ABC and let Y, Z be the intersections of Γ with the tangents from X with respect to $\mathcal{K}$. As X varies on Γ, what is the locus of the incenters of triangles XYZ?

Proposed by Cosmin Pohoata, Princeton University, USA

S262. Let a, b, c be the sides of a triangle and let m_a, m_b, m_c be the lengths of its medians. Prove that

$$a^2+b^2+c^2-ab-bc-ca\leq 4\left(m_a^2+m_b^2+m_c^2-m_am_b-m_bm_c-m_cm_a\right).$$

Proposed by Arkady Alt , San Jose, California, USA

S263. Prove that for all $n\geq 2$ and all $1\leq i\leq n$ we have

$$\sum_{j=1}^{n}(-1)^{n-j}\frac{\binom{n+j}{n}\binom{n}{n-j}}{i+j}=1.$$

Proposed by Marcel Chirita, Bucharest, Romania

S264. Let a, b, c, x, y, z be positive real numbers such that $ab+bc+ca=xy+yz+zx=1$. Prove that

$$a(y+z)+b(z+x)+c(x+y)\geq 2.$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania