S192. Let s, R, r and r_a, r_b, r_c be the semiperimeter, circumradius, inradius, and exradii of a triangle ABC. Prove that

$$s\sqrt{\frac{2}{R}} \leq \sqrt{r_a} + \sqrt{r_b} + \sqrt{r_c} \leq \frac{s}{\sqrt{r}}.$$

Proposed by Arkady Alt, San Jose, California, USA

First solution by Anthony Erb Lugo, San Juan, Puerto Rico

Let K denote the area of triangle ABC (with sides a, b and c). We have that

$$K = r_a(s-a) = rs = \sqrt{s(s-a)(s-b)(s-c)} = \frac{abc}{4R}$$

We start by proving the right hand side of the inequality, note that

$$K^2 = (r_a(s-a))(rs) = s(s-a)(s-b)(s-c)$$

or

$$r_a r = (s-b)(s-c) \implies \sqrt{r_a r} = \sqrt{(s-b)(s-c)}$$

Next, by the AM-GM inequality, we have

$$\sqrt{r_a r} = \sqrt{(s-b)(s-c)} \leq \frac{(s-b)+(s-c)}{2} = \frac{a}{2}$$

Applying this cyclically

$$\sqrt{r_a r} + \sqrt{r_b r} + \sqrt{r_c r} \leq \frac{a+b+c}{2} = s$$

Next, we divide by $\sqrt{r}$ on both sides, this ends the proof of the right hand side. Now we need to prove the left hand side

$$s\sqrt{\frac{2}{R}} \leq \sqrt{r_a} + \sqrt{r_b} + \sqrt{r_c}$$

We multiply both sides by $\sqrt{2R}$ so that the inequality is equivalent with

$$a+b+c \leq \sqrt{2r_a R} + \sqrt{2r_b R} + \sqrt{2r_a R}$$

Next, we recall the equality

$$r_a(s-a) = \frac{abc}{4R}$$

which is equivalent to

$$2r_a R = \frac{abc}{2(s-a)} = \frac{abc}{b+c-a}$$

Thus, applying the last equality cyclically, it is sufficient to prove that

$$a+b+c \leq \sqrt{\frac{abc}{b+c-a}} + \sqrt{\frac{abc}{a+c-b}} + \sqrt{\frac{abc}{a+b-c}}$$

which is the same as problem O181 from issue 1 of 2011.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Consider the triangles whose sides are the internal bisector of A, line AB, and the respective perpendiculars to AB through the incenter and the excenter opposite vertex A. It is well known that he feet