S190. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}\leq\frac{1}{9}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2.$$

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Solution by Anthony Erb Lugo, San Juan, Puerto Rico

By the AM-GM Inequality, we have

$$\frac{1}{2a^2+bc}=\frac{1}{a^2+a^2+bc}\leq\frac{1}{3a\sqrt[3]{abc}}$$

This implies that

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}\leq\frac{1}{3\sqrt[3]{abc}}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Thus, it is sufficient to prove that

$$\frac{1}{3\sqrt[3]{abc}}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\leq\frac{1}{9}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2$$

But this last inequality is equivalent to

$$\frac{3}{\sqrt[3]{abc}}\leq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

which follows directly from the AM-GM Inequality, so we are done.

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