S183. Let $a_0 \in (0,1)$ and $a_n = a_{n-1} - \frac{a_{n-1}^2}{2}$ for $n \geq 1$. Prove that for all $n \geq 1$,

$$\frac{n}{2} < \frac{1}{a_n} - \frac{1}{a_0} < \frac{n+1+\sqrt{n}}{2}.$$

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LHS. Note that $a_n \in (0,1)$ for any n. Indeed, by math induction, because the function $h(x) = x - \dfrac{x^2}{2}$ is increasing on $[0,1]$ and $a_0 \in (0,1)$ then the assumption $a_n \in (0,1)$ yields

$$a_{n+1} = h(a_n) \in (h(0), h(1)) = \left(0, \frac{1}{2}\right) \subset (0,1).$$

Furthermore,

$$\frac{1}{a_n} = \frac{1}{a_{n-1} - \frac{a_{n-1}^2}{2}} = \frac{1}{a_{n-1}} + \frac{1}{2}\frac{1}{1 - \frac{a_{n-1}}{2}}$$

$$\frac{1}{a_n} - \frac{1}{a_0} = \sum_{k=1}^{n}\left(\frac{1}{a_k} - \frac{1}{a_{k-1}}\right) = \sum_{k=1}^{n}\frac{1}{2 - a_{k-1}} > \sum_{k=1}^{n}\frac{1}{2} = \frac{n}{2}$$

RHS. We proceed by induction. For $n = 0$ clearly holds. Let's suppose it true for $1 \leq n \leq r$. For $n = r+1$ we have

$$\frac{1}{a_{n+1}} - \frac{1}{a_0} = \frac{1}{a_{n+1}} - \frac{1}{a_n} + \frac{1}{a_n} - \frac{1}{a_0} = \frac{1}{2 - a_n} + \frac{1}{a_n} - \frac{1}{a_0}$$

By using the induction hypotheses for $n \geq 1$

$$\frac{1}{2 - a_n} + \frac{1}{a_n} - \frac{1}{a_0} \leq \frac{1}{2 - a_n} + \frac{n+1+\sqrt{n}}{2} \leq \frac{n+2+\sqrt{n+1}}{2}$$

namely

$$a_n \leq 2\frac{\sqrt{n+1} - \sqrt{n}}{1 + \sqrt{n+1} - \sqrt{n}} = 2\frac{1}{\sqrt{n+1} + \sqrt{n}}\frac{1}{1 + \sqrt{n+1} - \sqrt{n}}$$

which is implied by

$$a_n \leq \frac{2}{2\sqrt{n+1}}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}\sqrt{n}} \tag{1}$$

so we need to show (1) for $n \geq 1$. By the LHS. we have

$$\frac{n}{2} \leq \frac{1}{a_n} - \frac{1}{a_0} \Rightarrow a_n \leq \frac{2a_0}{2 + na_0} \leq \frac{2}{2+n}$$

since the function $x/(1+cx)$ increases for $0 < x < 1$ if $c > 0$. Thus

$$a_n \leq \frac{2}{2+n} \leq \frac{1}{\sqrt{2}\sqrt{n}} \iff (n-2)^2 \geq 0$$

and we are done.

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