Senior problems

S181. Let a and b be positive real numbers such that

$$|a-2b| \le \frac{1}{\sqrt{a}} \quad \text{and} \quad |2a-b| \le \frac{1}{\sqrt{b}}.$$

Prove that $a+b \le 2$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S182. Let a, b, c be real numbers such that $a > b > c$. Prove that for each real number x the following inequality holds

$$\sum_{\text{cyc}} (x-a)^4(b-c) \ge \frac{1}{6}(a-b)(b-c)(a-c)[(a-b)^2+(b-c)^2+(c-a)^2].$$

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

S183. Let $a_0 \in (0,1)$ and $a_n = a_{n-1} - \frac{a_{n-1}^2}{2}$, $n \ge 1$. Prove that for all n,

$$\frac{n}{2} \le \frac{1}{a_n} - \frac{1}{a_0} < \frac{n+1+\sqrt{n}}{2}.$$

Proposed by Arkady Alt, San Jose, California, USA

S184. Let $H_n = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, $n \ge 2$. Prove that

$$e^{H_n} > \sqrt[n]{n!} \ge 2^{H_n}.$$

Proposed by Tigran Hakobyan, Vanadzor, Armenia

S185. Let A_1, A_2, A_3 be non-collinear points on parabola $x^2 = 4py$, $p > 0$, and let $B_1 = l_2 \cap l_3, B_2 = l_3 \cap l_1, B_3 = l_1 \cap l_2$ where l_1, l_2, l_3 are tangents to the parabola at points A_1, A_2, A_3, respectively. Prove that $\dfrac{[A_1A_2A_3]}{[B_1B_2B_3]}$ is a constant and find its value.

Proposed by Arkady Alt, San Jose, California, USA

S186. We wish to assign probabilities p_k, $k = 0,1,2,3$, to random variables X_1, X_2, and X_3 taking values in the set $\{0,1,2,3\}$ (some of them possibly with probability 0), such that the X_i, $i = 1,2,3$, will be identically distributed with $P(X_i = k) = p_k$, $k = 0,1,2,3$, and $X_1 + X_2 + X_3 = 3$. Prove that this is possible if and only if $p_2 + p_3 \le 1/3$, $p_1 = 1 - 2p_2 - 3p_3$, and $p_0 = p_2 + 2p_3$.

Proposed by Shai Covo, Kiryat-Ono, Israel