Olympiad problems

O85. Let a, b, c be non-negative real numbers such that $ab+bc+ca=1$. Prove that

$$4 \leq \left(\frac{1}{\sqrt{1+a^2}}+\frac{1}{\sqrt{1+b^2}}+\frac{1}{\sqrt{1+c^2}}\right)(a+b+c-abc).$$

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Calling $x=b+c$, $y=c+a$, $z=a+b$, we have $a+b+c=\frac{x+y+z}{2}$. Moreover, we may write $1+a^2=ab+bc+ca+a^2=(a+b)(c+a)=yz$ and similarly $1+b^2=zx$, $1+c^2=xy$. Furthermore, $(1-ab)=bc+ca=cz$, leading to $c-abc=c^2z=z(xy-1)=xyz-z$. Similarly, $a-abc=xyz-x$ and $b-abc=xyz-y$. Then,

$$a+b+c-abc=\frac{2(a+b+c)+(a-abc)+(b-abc)+(c-abc)}{3}=xyz,$$

and we may rewrite the proposed inequality as

$$4 \leq \left(\frac{1}{\sqrt{yz}}+\frac{1}{\sqrt{zx}}+\frac{1}{\sqrt{xy}}\right)xyz=\sqrt{xy}\sqrt{yz}+\sqrt{yz}\sqrt{zx}+\sqrt{zx}\sqrt{xy}.$$

Now, using the inequality between arithmetic and quadratic means, we have

$$\sqrt{yz}=\sqrt{1+a^2}=\frac{2}{\sqrt{3}}\sqrt{\frac{1+1+1+3a^2}{4}} \geq \frac{2}{\sqrt{3}}\frac{3+a\sqrt{3}}{4}=\frac{a+\sqrt{3}}{2},$$

with equality iff $a=\frac{1}{\sqrt{3}}$, and similarly for the other two products, leading to $\sqrt{zx}\sqrt{xy} \geq \frac{3+\sqrt{3}(b+c)+bc}{4}$, and similarly for the other two combinations. It then suffices to prove that

$$4 \leq \frac{9+2\sqrt{3}(a+b+c)+1}{4},$$

or equivalently, that $a+b+c \geq \sqrt{3}$. But this is true since, using the scalar product (Cauchy-Schwarz) inequality, we have

$$2=2(ab+bc+ca)=(a+b+c)^2-(a^2+b^2+c^2) \leq (a+b+c)^2-(ab+bc+ca),$$

and $(a+b+c)^2 \geq 2+(ab+bc+ca)=3$, with equality iff $a=b=c$. The result follows, and equality is reached if and only if $a=b=c=\frac{1}{\sqrt{3}}$.

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