Olympiad problems

O85. Let a, b, c be non-negative real numbers such that $ab+bc+ca=1$. Prove that

$$4 \leq \left(\frac{1}{\sqrt{1+a^2}}+\frac{1}{\sqrt{1+b^2}}+\frac{1}{\sqrt{1+c^2}}\right)(a+b+c-abc).$$

Proposed by Arkady Alt, San Jose, California, USA

O86. The sequence $\{x_n\}$ is defined by $x_1=1$, $x_2=3$ and $x_{n+1}=6x_n-x_{n-1}$ for all $n \geq 1$. Prove that $x_n+(-1)^n$ is a perfect square for all $n \geq 1$.

Proposed by Brian Bradie, Christopher Newport University, USA

O87. Let G be a graph with n vertices, $n \geq 5$. The edges of a graph are colored in two colors such that there are no monochromatic cycles of length $3, 4$, and 5. Prove that there are no more than $\left\lfloor \frac{n^2}{3} \right\rfloor$ edges in the graph.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

O88. Determine all pairs (z, n) such that

$$z+z^2+\ldots+z^n=n|z|,$$

where $z \in C$ and $|z| \in \mathbb{Z}_+$.

Proposed by Dorin Andrica, Babes-Bolyai University and Mihai Piticari, Campulung Moldovenesc, Romania

O89. Let P be an arbitrary point in the interior of a triangle ABC and let P' be its isogonal conjugate. Let I be the incenter of triangle ABC and let X, Y, Z be the midpoints of the small arcs BC, CA, AB. Denote by A_1, B_1, C_1 the intersections of lines AP, BP, CP with sides BC, CA, AB, respectively, and let A_2, B_2, C_2 be the midpoints of the segments IA_1, IB_1, IC_1. Prove that lines XA_2, YB_2, ZC_2 are concurrent on line IP'.

Proposed by Cosmin Pohoata, Tudor Vianu National College, Romania

O90. Find all positive integers n having at most four distinct prime divisors such that

$$n \mid 2^{\phi(n)}+3^{\phi(n)}+\ldots+n^{\phi(n)}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Gabriel Dospinescu, Ecole Normale Superieure, France