Olympiad problems

O295. Let a, b, c, x, y, z be positive real numbers such that $x+y+z=1$ and

$$2ab+2bc+2ca > a^2+b^2+c^2.$$

Prove that

$$a(x+3yz)+b(y+3xz)+c(z+3xy) \le \frac{2}{3}(a+b+c).$$

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Using $x+y+z=1$ we get

$$\begin{aligned} B &= 2(a+b+c) - 3\left[a(x+3yz)+b(y+3zx)+c(z+3xy)\right] \\ &= 2(a+b+c)(x+y+z)^2 - 3\sum_{cyc} a\left[x(x+y+z)+3yz\right] \\ &= 2au^2+avw+2bv^2+bwu+2cw^2+cuv, \end{aligned}$$

where $u=y-z$, $v=z-x$, and $w=x-y$. Replacing w by $-(u+v)$, we obtain further

$$B=(2c+2a-b)u^2+(5c-a-b)uv+(2b+2c-a)v^2.$$

Now the discriminant

$$(5c-a-b)^2-4(2c+2a-b)(2b+2c-a)=9(a^2+b^2+c^2-2ab-2bc-2ca)<0.$$

Also, $(2c+2a-b)+(2b+2c-a)=a+b+4c>0$. So both $2c+2a-b>0$ and $2b+2c-a>0$. Therefore, $B\ge 0$, completing the proof.

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