O274. Let a, b, c be positive integers such that a and b are relatively prime. Find the number of lattice points in

$$D = \{(x,y)|x,y \geq 0, bx + ay \leq abc\}.$$

Proposed by Arkady Alt, San Jose, California, USA

Solution by Li Zhou, Polk State College, USA

Let $X = (ac, 0)$ and $Y = (0, bc)$. Since the equation of XY is $y = b\left(c - \frac{x}{a}\right)$ and a, b are relatively prime, y is an integer if and only if $a|x$. Hence, the number of lattice points in the interior of segment XY is $c-1$. Then it is easy to see that the number of lattice points on the boundary of D is $B = ac + bc + c$. Let I be the the number of lattice points in the interior of D. By Pick's theorem, $I + \frac{1}{2}B - 1$ equals the area of D, which is $\frac{1}{2}abc^2$. Therefore, the number of lattice points in D is

$$I + B = \frac{1}{2}(abc^2 + B) + 1 = \frac{c(abc + a + b + 1)}{2} + 1.$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain.