Olympiad problems

O271. Let $(a_n)_{n\geq 0}$ be the sequence given by $a_0 = 0$, $a_1 = 2$ and $a_{n+2} = 6a_{n+1} - a_n$ for $n \geq 0$. Let $f(n)$ be the highest power of 2 that divides n. Prove that $f(a_n) = f(2n)$ for all $n \geq 0$.

Proposed by Albert Stadler, Herrliberg, Switzerland

O272. Let ABC be an acute triangle with orthocenter H and let X be a point in its plane. Let X_a, X_b, X_c be the reflections of X across AH, BH, CH, respectively. Prove that the circumcenters of triangles AHX_a, BXH_b, CXH_c are collinear.

Proposed by Michal Rolinek, Institute of Science and Technology, Vienna and Josef Tkadlec, Charles University, Prague

O273. Let P be a polygon with perimeter L. For a point X, denote by $f(X)$ the sum of the distances to the vertices of P. Prove that for any point X in the interior of P, $f(X) < \frac{n-1}{2}L$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O274. Let a, b, c be positive integers such that a and b are relatively prime. Find the number of lattice points in

$$D = \{(x, y) \mid \ x, y \geq 0,\ bx + ay \leq abc\}.$$

Proposed by Arkady Alt, San Jose, California, USA

O275. Let ABC be a triangle with circumcircle $\Gamma(O)$ and let ℓ be a line in the plane which intersects the lines BC, CA, AB at X, Y, Z, respectively. Let ℓ_A, ℓ_B, ℓ_C be the reflections of ℓ across BC, CA, AB, respectively. Furthermore, let M be the Miquel point of triangle ABC with respect to line ℓ.

a) Prove that lines ℓ_A, ℓ_B, ℓ_C determine a triangle whose incenter lies on the circumcircle of triangle ABC.

b) If S is the incenter from (a) and O_a, O_b, O_c denote the circumcenters of triangles AYZ, BZX, CXY, respectively, prove that the circumcircles of triangles SOO_a, SOO_b, SOO_c are concurrent at a second point, which lies on Γ.

Proposed by Cosmin Pohoata, Princeton University, USA

O276. For a prime p, let $S_1(p) = \{(a, b, c) \in \mathbf{Z}^3, \quad p | a^2b^2 + b^2c^2 + c^2a^2 + 1\}$ and $S_2(p) = \{(a, b, c) \in \mathbf{Z}^3, \quad p | a^2b^2c^2(a^2 + b^2 + c^2 + a^2b^2c^2)\}$. Find all p for which $S_1(p) \subset S_2(p)$.

Proposed by Titu Andreescu, University of Texas at Dallas and Gabriel Dospinescu, Ecole Normale Suprieure, Lyon