O260. Let p be a positive real number. Define a sequence $(a_n)_{n\geq 1}$ by $a_1 = 0$ and

$$a_n = \left\lfloor \frac{n+1}{2} \right\rfloor^p + a_{\lfloor \frac{n}{2} \rfloor}$$

for $n \geq 2$. Find the minimum of $\frac{a_n}{n^p-1}$ over all positive integers n.

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We observe that :

$$a_{2n} = \left\lfloor \frac{2n+1}{2} \right\rfloor^p + a_{\lfloor \frac{2n}{2} \rfloor} = n^p + a_n$$

$$a_{2n+1} = \left\lfloor \frac{2n+2}{2} \right\rfloor^p + a_{\lfloor \frac{2n+1}{2} \rfloor} = (n+1)^p + a_n$$

therefore we conclude from this two expressions that:

$$a_{2n+1} - a_{2n} = (n+1)^p - n^p$$

that can be rewriten :

$$a_{n+1} - a_n = \left(\frac{n}{2}+1\right)^p - \left(\frac{n}{2}\right)^p$$

from where we deduce that a_n is increasing and that for $n \geq 3$:

$$a_n = 1 + \sum_{k=2}^{n-1} \left(\left(\frac{k}{2}+1\right)^p - \left(\frac{k}{2}\right)^p \right)$$

The sequence $s_n = \frac{a_n}{n^p-1}$ is decreasing and the minimum of s_n over all positive integers n is attained for

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1 + \sum_{k=2}^{n-1} \left(\left(\frac{k}{2}+1\right)^p - \left(\frac{k}{2}\right)^p \right)}{n^p - 1}$$

or:

$$\frac{\sum_{k=2}^{n-1} \left(\left(\frac{k}{2}+1\right)^p - \left(\frac{k}{2}\right)^p \right)}{n^p - 1} = \frac{1}{2^p} \frac{\sum_{k=2}^{n-1} \left((k+2)^p - k^p\right)}{n^p - 1}$$

or:

$$\sum_{k=2}^{n-1} [(k+2)^p - k^p] = \sum_{k=2}^{n-1} [(k+2)^p - (k+1)^p + (k+1)^p - k^p]$$

$$= \sum_{k=2}^{n-1} [(k+2)^p - (k+1)^p] + \sum_{k=2}^{n-1} [(k+1)^p - k^p] = (n+1)^p - 3^p + n^p - 2^p$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sum_{k=2}^{n-1} [(k+2)^p - k^p]}{n^p - 1} = \lim_{n\to\infty} \frac{(n+1)^p - 3^p + n^p - 2^p}{n^p - 1}$$

$$= \lim_{n\to\infty} \frac{(n+1)^p + n^p}{n^p - 1} + \lim_{n\to\infty} \frac{\overbrace{-3^p - 2^p}^{Cste}}{n^p - 1} = \lim_{n\to\infty} \frac{(n+1)^p + n^p}{n^p - 1} + 0$$

$$= \lim_{n\to\infty} \frac{(n+1)^p + n^p}{n^p - 1} = \lim_{n\to\infty} \frac{(n+1)^p}{n^p - 1} + \lim_{n\to\infty} \frac{n^p}{n^p - 1}$$

$$= \lim_{n\to\infty} \frac{\frac{(n+1)^p}{n^p}}{\frac{n^p-1}{n^p}} + \lim_{n\to\infty} \frac{n^p}{n^p - 1} = \frac{\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^p}{1} + 1 = \frac{1}{1} + 1 = 2$$

which means that :

$$\lim_{n\to\infty} s_n = \frac{1}{2^{p-1}}$$

which is the minimum we look for. We are done.

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