Olympiad problems

O259. Solve in integers the equation $x^5 + 15xy + y^5 = 1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O260. Let p be a positive real number. Define a sequence $(a_n)_{n\geq 1}$ by $a_1 = 0$ and

$$a_n = \left\lfloor \frac{n+1}{2} \right\rfloor^p + a_{\lfloor \frac{n}{2} \rfloor}$$

for $n \geq 2$. Find the minimum of $\frac{a_n}{n^p - 1}$ over integers $n \geq 2$.

Proposed by Arkady Alt, San Jose, California, USA

O261. Find all positive integers n for which

$$\sigma(n) - \phi(n) \leq 4\sqrt{n},$$

where $\sigma(n)$ is the sum of positive divisors of n and ϕ is Euler's totient function.

Proposed by Albert Stadler, Buchenrain, Herrliberg, Switzerland

O262. Let G be a finite directed graph. Prove that there is an integer N such that no matter how one chooses directions for edges of an undirected graph G' with $|G'| \geq N$, one will always get a copy of G as an induced subgraph of G'.

Proposed by Cosmin Pohoata, Princeton University, USA

O263. Let $n \geq 3$ be an integer. Consider a convex n-gon $A_1 \dots A_n$ for which there is a point P in its interior such that $\angle A_i P A_{i+1} = \frac{2\pi}{n}$ for all $i \in [1, n-1]$. Prove that P is the point which minimizes the sum of distances to the vertices of the n-gon.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O264. Let $p > 3$ be a prime. Prove that $2^{p-1} \equiv 1 \pmod{p^2}$ if and only if the numerator of

$$\frac{1}{2} + \frac{1}{3}\left(1 + \frac{1}{2}\right) + \cdots + \frac{1}{\frac{p-1}{2}}\left(1 + \frac{1}{2} + \cdots + \frac{1}{\frac{p-3}{2}}\right)$$

is a multiple of p.

Proposed by Gabriel Dospinescu, Lyon, France