J93. Let a and b be positive real numbers. Prove that

$$\frac{a^6+b^6}{a^4+b^4} \geq \frac{a^4+b^4}{a^3+b^3} \cdot \frac{a^2+b^2}{a+b}.$$

Proposed by Arkady Alt, San Jose, California, USA

First solution by Nguyen Manh Dung, Hanoi University of Science, Vietnam

The above inequality is equivalent to

$$(a^6+b^6)(a^3+b^3)(a+b) \geq (a^4+b^4)^2(a^2+b^2).$$

Using the Cauchy-Scharz inequality, we have

$$(a^6+b^6)(a^2+b^2) \geq (a^4+b^4)^2$$

$$(a^3+b^3)(a+b) \geq (a^2+b^2)^2.$$

Multiplying these inequalities, the conclusion follows. Equality occurs when $a=b$.

Second solution by Shamil Asgarli, Canada

After multiplying out the right side the inequality becomes

$$\frac{a^6+b^6}{a^4+b^4} \geq \frac{a^6+a^4b^2+b^4a^2+b^6}{a^4+a^3b+b^3a+b^4}$$

or

$$\frac{a^6+b^6}{a^4+b^4} \geq \frac{a^6+b^6+a^2b^2(a^2+b^2)}{a^4+b^4+ab(a^2+b^2)}.$$

To simplify calculations let use make the following substitutions

$$a^6+b^6=x$$

$$a^4+b^4=y$$

$$ab=z$$

$$a^2+b^2=t.$$

Our inequality transforms to

$$\frac{x}{y} \geq \frac{x+z^2t}{y+zt}$$

or after some algebra $x \geq yz$. If we back substitute we obtain

$$a^6 + b^6 \geq (a^4 + b^4)ab$$

which is exactly the same as

$$(a^5 - b^5)(a - b) \geq 0$$

and thus we are done. Equality occurs when $a = b$.

Third solution by An Zhen-ping, China

Let us start with the following observation

$$(a^6 + b^6)(a^3 + b^3) - (a^5 + b^5)(a^4 + b^4) = a^3 b^3 (a + b)(a - b)^2 \geq 0.$$

The last observation implies that

$$\frac{(a^6 + b^6)}{(a^5 + b^5)} \geq \frac{(a^4 + b^4)}{(a^3 + b^3)}.$$

By the same token we observe that

$$\frac{(a^5 + b^5)}{(a^4 + b^4)} \geq \frac{(a^2 + b^2)}{(a + b)}.$$

The final step is to multiply the last two inequalities to obtain the desired result.

Also solved by Andrea Munaro, Italy; Brian Bradie, Newport News, VA; Daniel Lasaosa, Universidad Publica de Navarra, Spain; John T. Robinson, Yorktown Heights, NY, USA; Michel Bataille, France; Oleh Faynshteyn, Leipzig, Germany; Oles Dobosevych, Ukraine.