Junior problems

J85. Let a and b be positive real numbers. Prove that

$$\sqrt[3]{\frac{(a+b)(a^2+b^2)}{4}} \geq \sqrt{\frac{a^2+ab+b^2}{3}}.$$

Proposed by Arkady Alt, San Jose, California, USA

First solution by Ivanov Andrey, Chisinau, Moldova

Raising to the sixth power we obtain

$$\left(\frac{(a+b)(a^2+b^2)}{4}\right)^2 \geq \left(\frac{a^2+ab+b^2}{3}\right)^3$$

or

$$\frac{(a+b)^2(a^2+b^2)^2}{16} - \frac{(a^2+ab+b^2)^3}{27} \geq 0.$$

Algebraic manipulations yeild

$$27((a-b)^2+4ab)((a-b)^2+2ab)^2 - 16((a-b)^2+3ab)^3 \geq 0.$$

Let $(a-b)^2 = x \geq 0$ and $ab = y \geq 0$. Then we have:

$$27(x+4y)(x+2y)^2 - 16(x+3y)^3 \geq 0$$

which in turn yeilds

$$x(11x^2+72xy+108y^2) \geq 0.$$

Because $x, y \geq 0$ the last inequality is true. Equality holds if $x = 0$ or $a = b$.

Second solution by Manh Dung Nguyen, Vietnam

The above inequality is equivalent to

$$\begin{aligned}
&\frac{3(a^2+b^2)}{2(a^2+ab+b^2)} \geq \frac{2\sqrt{a^2+ab+b^2}}{\sqrt{3}(a+b)} \\
\Leftrightarrow\ &\frac{3(a^2+b^2)}{2(a^2+ab+b^2)} - 1 \geq \frac{2\sqrt{a^2+ab+b^2}}{\sqrt{3}(a+b)} - 1 \\
\Leftrightarrow\ &\frac{(a-b)^2}{2(a^2+ab+b^2)} \geq \frac{(a-b)^2}{\sqrt{3}(a+b)\left(2\sqrt{a^2+ab+b^2}+\sqrt{3}(a+b)\right)}.
\end{aligned}$$

So we need to prove that

$$\sqrt{3}(a+b)\left(2\sqrt{a^2+ab+b^2}+\sqrt{3}(a+b)\right) \geq \sqrt{3}(a+b)2\sqrt{3}(a+b)$$
$$= 6(a+b)^2 > 2(a^2+ab+b^2).$$

And thus we are done. Equality holds if and only if $a = b$.

Third solution by Oleh Faynshteyn, Leipzig, Germany

Raise the inequality to the sixth power to obtain the equivalent inequality

$$\frac{(a+b)^2(a^2+b^2)^2}{16} \geq \frac{(a^2+ab+b^2)^3}{27}$$

or after transformations

$$11a^6+6a^5b-15a^4b^2-4a^3b^3-15a^2b^4+6ab^5+11b^6 \geq 0,$$

or

$$11\left(\frac{a}{b}\right)^6+6\left(\frac{a}{b}\right)^5-15\left(\frac{a}{b}\right)^4-4\left(\frac{a}{b}\right)^3-15\left(\frac{a}{b}\right)^2+6\left(\frac{a}{b}\right)+11 \geq 0.$$

Substituting $\frac{a}{b}$ by t we obtain

$$11t^6+6t^5-15t^4-4t^3-15t^2+6t+11 = (t-1)^2(11t^4+28t^3+30t^2+28t+11) \geq 0$$

and thus we are done with equality when $t = 1$ or $a = b$.

Fourth solution by Ovidiu Furdui, Cluj, Romania

The inequality to prove is equivalent to

$$\sqrt[3]{\frac{(a/b+1)(a^2/b^2+1)}{4}} \geq \sqrt{\frac{a^2/b^2+a/b+1}{3}}.$$

Thus, it suffices to prove that for all $x > 0$ the following inequality holds

$$\sqrt[3]{\frac{(x+1)(x^2+1)}{4}} \geq \sqrt{\frac{x^2+x+1}{3}}.$$

Straight forward calculations show that the preceding inequality is equivalent to

$$(11x^4+28x^3+30x^2+28x+11)(x-1)^2 \geq 0,$$

which holds for all $x > 0$, and the problem is solved.

Also solved by Andrea Munaro, Italy; Brian Bradie, Newport News, VA; Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Publica de Navarra, Spain; G.R.A.20 Math Problems Group, Roma, Italy; Michel Bataille, France; Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; Roberto Bosch Cabrera, Cuba.