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838. *Proposed by Arkady Alt, San Jose, CA.*

Show that in any acute triangle ΔABC with sides a, b and c, the following inequality is true:

$$27 \leq (a+b+c)^2 \left(\frac{1}{a^2+b^2-c^2} + \frac{1}{b^2+c^2-a^2} + \frac{1}{c^2+a^2-b^2} \right)$$

Solution proposed by G.R.A.20 Problem Solving Group, Roma, Italy.

Since $c^2 = a^2 + b^2 - 2ab\cos\gamma$ then

$$\frac{1}{a^2+b^2-c^2} = \frac{1}{2ab\cos\gamma} = \frac{\tan\gamma}{4A}$$

where A is the triangle's area. Simmetrically

$$\frac{1}{b^2+c^2-a^2} = \frac{\tan\alpha}{4A} \quad \text{and} \quad \frac{1}{c^2+a^2-b^2} = \frac{\tan\beta}{4A}.$$

After transforming and rearranging the terms, the inequality becomes

$$\sqrt{27} \cdot \frac{A}{s^2} \leq \frac{\tan\alpha + \tan\beta + \tan\gamma}{\sqrt{27}}$$

where s is the triangle's semiperimeter. Now it suffices to prove that

$$\frac{A}{s^2} \leq \frac{1}{\sqrt{27}} \quad \text{and} \quad \sqrt{27} \leq \tan\alpha + \tan\beta + \tan\gamma.$$

First inequality: by Heron's formula $A = \sqrt{s(s-a)(s-b)(s-c)}$ and by applying AGM inequality we obtain

$$\frac{A}{s^2} = \sqrt{\left(1-\frac{a}{s}\right)\left(1-\frac{b}{s}\right)\left(1-\frac{c}{s}\right)} \leq \sqrt{\left(\frac{1}{3}\left(1-\frac{a}{s}+1-\frac{b}{s}+1-\frac{c}{s}\right)\right)^3} = \frac{1}{\sqrt{27}}.$$

Second inequality: we note that

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

therefore, since $\tan\gamma = -\tan(\alpha+\beta)$, we find that

$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma.$$

Since the triangle is acute then $\tan\alpha$, $\tan\beta$ and $\tan\gamma$ are positive and by applying AGM inequality we obtain

$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma \le \left(\frac{\tan\alpha + \tan\beta + \tan\gamma}{3}\right)^3$$

which means that

$$\sqrt{27} \le \tan\alpha + \tan\beta + \tan\gamma.$$

□