SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015: 41(5), p. 212–215.

4041. *Proposed by Arkady Alt.*

Let a, b and c be the side lengths of a triangle ABC. Let AA', BB' and CC' be the heights of the triangle and let $a_p = B'C', b_p = C'A'$ and $c_p = A'B'$ be the sides of the orthic triangle. Prove that:

a) $a^2(b_p + c_p) + b^2(c_p + a_p) + c^2(a_p + b_p) = 3abc$;

b) $a_p + b_p + c_p \leq s$, where s is the semiperimeter of ABC.

We received 15 correct solutions and present the solution by Michel Bataille.

We show (a) and (b) in the case when ΔABC has no obtuse angle and provide a counter-example in the opposite case.

First, suppose that angles A, B, C are acute. Since $\Delta AB'B$ is right-angled with $\angle AB'B = 90^\circ$, we have $AB' = c \cdot \cos A$. Similarly, $AC' = b \cdot \cos A$, and it follows that

$$\begin{aligned} B'C'^2 &= c^2 \cos^2 A + b^2 \cos^2 A - 2bc \cos^3 A \\ &= (c^2 + b^2 - 2bc \cos A) \cos^2 A = a^2 \cos^2 A \end{aligned}$$

and so $a_p = B'C' = a \cos A$. In a similar way, we obtain $b_p = A'C' = b \cos B$ and $c_p = A'B' = c \cos C$.

Now we calculate $X = a^2(b_p + c_p) + b^2(c_p + a_p) + c^2(a_p + b_p)$ as follows:

$$\begin{aligned} X &= a^2 b \cos B + a^2 c \cos C + b^2 c \cos C + b^2 a \cos A + bc^2 \cos B + c^2 a \cos A \\ &= ab(a \cos B + b \cos A) + bc(b \cos C + c \cos B) + ca(c \cos A + a \cos C) \\ &= abc + bca + cab = 3abc, \end{aligned}$$

as desired. Denoting by r and R the inradius and the circumradius of ΔABC and using the Law of Sines, we get

$$\begin{aligned} a_p + b_p + c_p &= a \cos A + b \cos B + c \cos C \\ &= R(\sin 2A + \sin 2B + \sin 2C) \\ &= 4R \sin A \sin B \sin C \\ &= 4R \cdot \frac{abc}{8R^3} = \frac{4rRs}{2R^2} = s \cdot \frac{2r}{R} \end{aligned}$$

and the result $a_p + b_p + c_p \leq s$ follows from Euler's inequality $2r \leq R$.

If ΔABC is right-angled, say $\angle BAC = 90°$, results (a) and (b) continue to hold if we take, as is natural, $a_p = 0$, $b_p = c_p = h$, where $h = AA'$. Indeed, we have $3abc = 3a \cdot ah = 3a^2h$ and

$$\begin{aligned} a^2(b_p + c_p) + b^2(c_p + a_p) + c^2(a_p + b_p) &= a^2 \cdot 2h + b^2 \cdot h + c^2 \cdot h \\ &= h(b^2 + c^2 + 2a^2) = 3a^2h. \end{aligned}$$

Also, the inequality $a_p + b_p + c_p \le s$ rewrites as $4h \le a + b + c$ or $4bc \le a^2 + a(b + c)$. Since $b + c \ge 2\sqrt{bc}$ and $a^2 = b^2 + c^2 \ge 2bc$, we have

$$a^2 + a(b + c) \ge 2bc + 2\sqrt{2bc} \cdot 2\sqrt{bc} = (2 + 2\sqrt{2})bc \ge 4bc.$$

None of these results is correct, however, if an angle of ΔABC is obtuse, as the following example shows. Consider a triangle ABC with $\angle BAC = 120°$ and $AB = AC$. Then $b = c$, $a = c\sqrt{3}$, and $a_p = b_p = c_p = \frac{a}{2} = \frac{c\sqrt{3}}{2}$. One easily finds that $3abc = 3c^3\sqrt{3}$, while

$$a^2(b_p + c_p) + b^2(c_p + a_p) + c^2(a_p + b_p) = 5c^3\sqrt{3}.$$

Also,

$$a_p + b_p + c_p = \frac{3c\sqrt{3}}{2} > \frac{(2 + \sqrt{3})c}{2} = s.$$

4042. *Proposed by Leonard Giugiuc and Diana Trailescu.*

Let a, b and c be real numbers in $[0, \pi/2]$ such that $a + b + c = \pi$. Prove the inequality

$$2\sqrt{2} \sin\frac{a}{2} \sin\frac{b}{2} \sin\frac{c}{2} \ge \sqrt{\cos a \cos b \cos c}.$$

We received 14 correct solutions. We present the solution by Scott Brown. Similar solutions came from Arslanagić Šefket, Michel Bataille, Andrea Fanchini, and John Heuvel.

In [1] and [2] respectively, we find the identities

$$\sin\frac{a}{2} \sin\frac{b}{2} \sin\frac{c}{2} = \frac{r}{4R} \qquad (1)$$

and

$$\cos a \cos b \cos c = \frac{s^2 - 4R^2 - 4Rr - r^2}{4R^2}, \qquad (2)$$

where R, r, and s are the circumradius, inradius, and semiperimeter of the triangle. We square both sides of the original inequality to obtain the equivalent statement

$$8 \sin^2\frac{a}{2} \sin^2\frac{b}{2} \sin^2\frac{c}{2} \le \cos a \cos b \cos c,$$