Editor's comment. The above proposer's solution is the only solution that did not make use of Bertrand's postulate.

Solution 2, by Joseph DiMuro, expanded by the editor.

Note that $c > a$ and $c > b$ since $c! > b^b > b!$. Suppose that p is a prime divisor of b. Then p must divide $b!$, b^b and $c!$, so that p must divide $a!$ and $p \leq a$. Thus, if $a = 1$, then $b = 1$ and we get the solution $(a, b, c) = (1, 1, 2)$. If $a = 2$, then $b \neq 1$ and the only prime divisor of b is 2. But then b^b is a multiple of 4 and $c! \equiv 2 \pmod 4$. The only possibility is $(a, b, c) = (2, 2, 3)$.

Suppose, if possible, that $a \geq 3$; let q be the largest prime that does not exceed a. Then, by Bertrand's postulate that when $m \geq 2$ there is always a prime between m and $2m$, $a < 2q$, so that q^2 cannot divide $a!$. However, since q divides a, it must divide $c!$ and hence divide b. Since $a \geq 3$, $a!$ and $c!$ are both even as is b. Because $b \neq 2$, we must have that $c \geq b \geq 2q$ (whether $q = 2$ or q is odd). Hence q^2 divides $c!$ and b^b and so must divide $a!$, yielding a contradiction.

Therefore, the sole solutions are $(a, b, c) = (1, 1, 2), (2, 2, 3)$.

3837. *Proposed by Arkady Alt.*

Let $(u_n)_{n \geq 0}$ be a sequence defined recursively by

$$u_{n+1} = \frac{u_n + u_{n-1} + u_{n-2} + u_{n-3}}{4},$$

for $n \geq 3$. Determine $\lim\limits_{n \to \infty} u_n$ in terms of u_0, u_1, u_2, u_3.

Solved by AN-anduud Problem Solving Group; R. Barbara; M. Bataille; M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal; P. Deiermann; J. DiMuro; O. Kouba; K. Lewis; Á. Plaza; C. R. Pranesachar; D. Smith; D. Stone and J. Hawkins; R. Zarnowski; and the proposer. We present 2 solutions.

Solution 1, by Joseph DiMuro.

We prove that $\lim\limits_{n \to \infty} u_n = \frac{1}{10} u_0 + \frac{2}{10} u_1 + \frac{3}{10} u_2 + \frac{4}{10} u_3$. The proof is by visual aid.

Put 10 water glasses on a table. Pour u_0 mL of water into one glass. Pour u_1 mL of water into each of 2 glasses, pour u_2 mL into each of 3 glasses, and pour u_3 mL into each of the remaining 4 glasses. Put the glasses into groups based on the amount of water in each glass. (So, the lone glass with u_0 mL is in a group by itself, the 2 glasses with u_1 mL form another group, and so on.)

Now, perform the following operation repeatedly: take one glass from each group. Pour water between those four glasses until they all have the same amount. Then put those four glasses back on the table as a new group. (Each of the old groups will have one fewer glass than before.)

After performing this operation once, you will have 1 glass with u_1 mL, 2 glasses with u_2 mL, 3 glasses with u_3 mL, and 4 glasses with u_4 mL. After performing this operation a second time, you will have 1 glass with u_2 mL, 2 glasses with u_3 mL, 3 glasses with u_4 mL, and 4 glasses with u_5 mL. And so on.

The amount of water in each glass will gradually approach $\lim\limits_{n\to\infty} u_n$. Therefore, $\lim\limits_{n\to\infty} u_n$ must be equal to the average amount of water per glass at the start:

$$\lim_{n\to\infty} u_n = \frac{1}{10}u_0 + \frac{2}{10}u_1 + \frac{3}{10}u_2 + \frac{4}{10}u_3.$$

Editor's comment. This solution, as well the argument above, generalize to any sequence where u_n is defined to be the average of the k previous terms. This solution does assume that $\lim_{n\to\infty} u_n$ exists. Its existence can be proven using the roots of the characteristic polynomial for the recurrence relation, as in the next solution.

Solution 2, by Michel Bataille.

The characteristic equation of the sequence $(u_n)_{n\geq 0}$ is

$$4x^4 - x^3 - x^2 - x - 1 = 0,$$

that is,

$$(x-1)(4x^3 + 3x^2 + 2x + 1) = 0.$$

The function $f : x \mapsto 4x^3 + 3x^2 + 2x + 1$ is continuous and strictly increasing on $\mathbb{R}$ with $f(\mathbb{R}) = \mathbb{R}$, so the equation $f(x) = 0$ has a unique real solution, say r. Noticing that $f(-1) = -2 < 0$ and $f(-\frac{1}{4}) > 0$, we see that

$$-1 < r < -\frac{1}{4} \qquad (1)$$

.

The non real solutions to $f(x) = 0$ are two complex conjugates z_0 and $\overline{z_0}$ and since $r \cdot z_0 \cdot \overline{z_0} = -\frac{1}{4}$, we have $|z_0|^2 = \frac{1}{4|r|}$, hence $|z_0| < 1$ since by (1), $|r| > \frac{1}{4}$.

From the list $1, r, z_0, \overline{z_0}$ of the roots of the characteristic equation, we deduce the form of u_n:

$$u_n = \alpha_1 + \alpha_2 r^n + \alpha_3 z_0^n + \alpha_4 \overline{z_0}^n$$

where the α_j are independent of n and determined from u_0, u_1, u_2, u_3.

Since $|r| < 1$ and $|z_0| = |\overline{z_0}| < 1$, we have $\lim_{n\to\infty} r^n = \lim_{n\to\infty} z_0^n = \lim_{n\to\infty} \overline{z_0}^n = 0$ so that $\lim\limits_{n\to\infty} u_n = \alpha_1$.

Now, the following relations hold

$$u_0 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \quad u_1 = \alpha_1 + \alpha_2 r + \alpha_3 z_0 + \alpha_4 \overline{z_0}, \quad u_2 = \alpha_1 + \alpha_2 r^2 + \alpha_3 z_0^2 + \alpha_4 \overline{z_0}^2$$

and

$$u_3 = \alpha_1 + \alpha_2 r^3 + \alpha_3 z_0^3 + \alpha_4 \overline{z_0}^3.$$

Since $f(r) = f(z_0) = f(\overline{z_0}) = 0$, we obtain

$$u_0 + 2u_1 + 3u_2 + 4u_3 = 10\alpha_1 + \alpha_2 f(r) + \alpha_3 f(z_0) + \alpha_4 f(\overline{z_0}) = 10\alpha_1$$

and we conclude
$$\lim_{n\to\infty} u_n = \frac{u_0 + 2u_1 + 3u_2 + 4u_3}{10}.$$

Editor's comment. Perfetti pointed out that this result appeared in *"On the Solutions of Linear Mean Recurrences"*, American Mathematical Monthly, 121 (6).

3838. *Proposed by Jung In Lee.*

Prove that there are no triplets (a, b, c) of distinct positive integers that satisfy the conditions:

- $a + b$ divides c^2, $b + c$ divides a^2, $c + a$ divides b^2, and
- the number of distinct prime factors of abc is at most 2.

Solved by M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal; J. DiMuro; S. Malikić; and the proposer. We present the proposer's solution.

Suppose (a, b, c) is a triplet of distinct positive integers satisfying the given conditions. Let $a = p^{x_1}q^{y_1}$, $b = p^{x_2}q^{y_2}$ and $c = p^{x_3}q^{y_3}$, where p and q are distinct prime numbers and x_i and y_i are nonnegative integers for $i = 1, 2, 3$. Let $i, j, k \in \{1, 2, 3\}$ such that $i \neq j \neq k \neq i$. We consider two cases separately.

Case 1. Suppose $x_i > x_j$ and $y_i > y_j$. Then we have
$$p^{x_j}q^{y_j}(p^{x_i-x_j}q^{y_i-y_j} + 1) = p^{x_i}q^{y_i} + p^{x_j}q^{y_j},$$
which divides $p^{2x_k}q^{2y_k}$. So
$$p^{x_i-x_j}q^{y_i-y_j} + 1 | p^{2x_k}q^{2y_k},$$
which is impossible since $(p^{x_i-x_j}q^{y_i-y_j} + 1, p^{2x_k}q^{2y_k}) = 1$.

Case 2. Suppose $x_i > x_j$ and $y_i < y_j$. Then we have
$$p^{x_j}q^{y_j}(p^{x_i-x_j} + q^{y_j-y_i}) = p^{x_i}q^{y_i} + p^{x_j}q^{y_j},$$
which divides $p^{2x_k}q^{2y_k}$. So
$$p^{x_i-x_j} + q^{y_j-y_i} | p^{2x_k}q^{2y_k},$$
which is impossible since $(p^{x_i-x_j} + q^{y_j-y_i}, p^{2x_k}q^{2y_k}) = 1$.

By cases 1 and 2, we have $x_i = x_j$ or $y_i = y_j$. It follows that either two or more of the statements $x_1 = x_2$, $x_2 = x_3$ and $x_3 = x_1$ are true or two or more of the statements $y_1 = y_2$, $y_2 = y_3$ and $y_3 = y_1$ are true. Hence $x_1 = x_2 = x_3$ or $y_1 = y_2 = y_3$. Without loss of generality, we assume that $x_1 = x_2 = x_3 = x$. Since the given conditions are homogenous in a, b and c, which are distinct, we may assume that $y_1 > y_2 > y_3$. Then
$$p^xq^{y_1} + p^xq^{y_2} = p^xq^{y_2}(q^{y_1-y_2} + 1),$$