solutions were received. The corrected version of the problem will appear in a future problem set.]

3688. [2011 : 540, 542] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let $T_n(x)$ be the Chebyshev polynomial of the first kind defined by the recurrence $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for $n \geq 1$ and the initial conditions $T_0(x) = 1$ and $T_1(x) = x$. Find all positive integers n such that

$$T_n(x) \ \geq \ (2^{n-2}+1)x^n - 2^{n-2}x^{n-1}\,, \quad x \in [1, \infty)\,.$$

[*Ed.: Note problem* **3585** *was originally printed with the wrong inequality.*]

Solution by Michel Bataille, Rouen, France.

For $n \geq 1$, let $P_n(x) = (2^{n-2}+1)x^n - 2^{n-2}x^{n-1}$. We will show that $T_n(x) \geq P_n(x)$ holds for all $x \in [1, \infty)$ if and only if $n \in \{2, 3, 4, 5, 6, 7\}$.

First, we notice that since $T_1(x) = x$ and $P_1(x) = \frac{3}{2}x - \frac{1}{2}$ then $T_1(x) < P_1(x)$ for $x \in (1, \infty)$, so we may assume $n \geq 2$ in what follows. It is well-known that $T_n(\cos\theta) = \cos(n\theta)$ for $\theta \in \mathbb{R}$. Using the fact that $\cos(n\theta)$ is the real part of $(\cos\theta + i\sin\theta)^n$ and the binomial theorem, it is readily obtained that

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} x^{n-2k}(x^2-1)^k.$$

It follows that

$$T_n(x) - P_n(x) = \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k} x^{n-2k}(x^2-1)^k - 2^{n-2}x^{n-1}(x-1) = x^{n-1}(x-1)\delta_n(x),$$

where

$$\delta_n(x) = \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k} \left(1 + \frac{1}{x}\right)^k \left(1 - \frac{1}{x}\right)^{k-1} - 2^{n-2}.$$

Therefore, $T_n(x) - P_n(x)$ has the same sign as $\delta_n(x)$ for $x > 1$. By induction, it is easy to show that $\delta_n(1) < 0$ for all $n \geq 8$ and by continuity, $\delta_n(1) < 0$ for $x > 1$ sufficiently close to 1. Thus, $T_n(x) \geq P_n(x)$ for all $x \in [1, \infty)$ can hold only if $n \in \{2, 3, 4, 5, 6, 7\}$.

Now, since

$$x\delta_2(x) = 1, \qquad x\delta_3(x) = x + 3,$$
$$x^3\delta_4(x) = x(3x^2-1) + (7x^2-1), \qquad x^3\delta_5(x) = x(7x^2-5) + (15x^2-5),$$

and

$$\begin{aligned} x^5\delta_6(x) &= 15x^5 + 31x^4 - 17x^3 - 17x^2 + x + 1 \\ &\geq 46x^4 - 17x^3 - 17x^2 + x + 1 = 17x^3(x-1) + x^2(29x^2-17) + x + 1, \end{aligned}$$

then $\delta_n(x) > 0$ for all $x > 1$ and $n = 2, 3, 4, 5, 6$. Finally, if $n = 7$, we obtain $x^5\delta_7(x) = \phi(x)$ where $\phi(x) = 31x^5 + 63x^4 - 49x^3 - 49x^2 + 7x + 7$. Since $\phi'(x) = 155x^4 + 252x^3 - 147x^2 - 98x + 7 > 0$ for $x \geq 1$, we have $\phi(x) > \phi(1) > 0$ for $x > 1$ and $\delta_7(x) > 0$ for all $x > 1$ again. This completes the proof.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer.

3689. [2011 : 540, 543] *Proposed by Ivaylo Kortezov, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria.*

In a group of n people, each one has a different book. We say that a pair of people performs a *swap* if they exchange the books they currently have. Find the least possible number $E(n)$ of swaps such that each pair of people has performed at least one swap and at the end each person has the book he or she had at the start.

Solution by M. A. Prasad, India; expanded slightly by the editor.

We show that

$$E(n) = \begin{cases} \frac{n(n-1)}{2}, & \text{if } n \equiv 0, 1 \pmod 4 \\ \frac{n(n-1)}{2} + 1, & \text{if } n \equiv 2, 3 \pmod 4 . \end{cases}$$

To avoid triviality, we assume that $n \geq 2$. Since there are $\binom{n}{2}$ pairs of people $E(n) \geq \binom{n}{2} = \frac{n(n-1)}{2}$. Furthermore, since everyone gets his/her book back at the end, E(n) must be even.

[Ed.: *Using the terminology and well known facts from the theory of permutation groups, $E(n)$ is the minimum number of transpositions (2-cycles) performed on the set $\{1, 2, 3, \ldots, n\}$ such that the product of which yields the identity permutation σ, and every transposition (i, j) must appear at least once in the product for all $i, j = 1, 2, \ldots, n$ with $i \neq j$. It is well known that a transposition is odd and σ is even. Hence there must be an even number of transpositions in the product.*]

Therefore, if $\frac{n(n-1)}{2}$ is odd, then $E(n) \geq \frac{n(n-1)}{2} + 1$.

Clearly $E(2) = 2$ and it is easy to see that $E(3) = 4$. We label the n people by $1, 2, \ldots, n$ and for $i, j = 1, 2, \ldots, n$ with $i \neq j$ we use (i, j) to denote the swap between i and j. For $n = 4$, the sequence of swaps $(1, 2)$, $(1, 3)$, $(2, 4)$, $(1, 4)$, $(2, 3)$, $(3, 4)$ (performed from left to right) shows that $E(4) = 6$ and for $n = 5$, the sequence $(1, 5)$, $(1, 2)$, $(2, 5)$, $(3, 5)$, $(3, 4)$, $(4, 5)$, $(2, 3)$, $(1, 4)$, $(1, 3)$, $(2, 4)$ shows that $E(5) = 10$. We now proceed by induction to prove our claim.

Suppose that $E(n) = \frac{n(n-1)}{2} + \mathcal{E}$ for some $n \geq 5$ where $\mathcal{E} = 0$ or 1 depending on whether $\frac{n(n-1)}{2}$ is even or odd. We show that

$$\begin{aligned} E(n+4) &= \frac{n(n-1)}{2} + \mathcal{E} + 4n + 6 \\ &= \frac{(n+4)(n+3)}{2} + \mathcal{E}. \end{aligned}$$

We denote the $n + 4$ books by $b_1, b_2, \ldots, b_{n+4}$ and break up the swaps into six steps as follows: