Let s be the semiperimeter. We need the following known results.

$$\sum_{\text{cyclic}} r_a = 4R + r, \tag{2}$$

$$\sum_{\text{cyclic}} r_a^2 = (4R + r)^2 - 2s^2, \tag{3}$$

$$\sum_{\text{cyclic}} a^2 = 2\left(s^2 - r^2 - 4Rr\right). \tag{4}$$

Formulas (2) and (3) appear on p.61 (items 99, 103) and formula (4) on p.52 (item 5) [1]. With these formulas, we can get

$$\begin{aligned}\frac{(r_a + r_b + r_c)^2}{\left(a^2 + r_a^2\right) + \left(b^2 + r_b^2\right) + \left(c^2 + r_c^2\right)} &= \frac{(4R + r)^2}{(4R + r)^2 - 2s^2 + 2\left(s^2 - r^2 - 4Rr\right)}\\ &= \frac{(4R + r)^2}{(4R + r)(4R - r)} = \frac{4R + r}{4R - r},\end{aligned}$$

and this with (1) completes the solution.

Also solved by JOE HOWARD, Portales, NM, USA; OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer.

Geupel and the proposer note that the identity $a^2 + b^2 + c^2 + r_a^2 + r_b^2 + r_c^2 = 16R^2 - r^2$, *used in the last step of the featured solution, is interesting on its own.*

References

[1] D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989.

3571. [2010 : 397, 399] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let $n \geq 1$ be an integer. Among all increasing arithmetic progressions x_1, x_2, ..., x_n such that $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, find the progression with the greatest common difference d.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Since the case $n = 1$ is degenerate, let us assume $n > 1$. An arithmetic progression $x_1, x_2, \ldots, x_n$ with common difference $d > 0$ has the property that $1 = x_1^2 + x_2^2 + \cdots + x_n^2$ if and only if

$$1 = \sum_{k=0}^{n-1}(x_1 + kd)^2 = nx_1^2 + 2d\left(\sum_{k=0}^{n-1} k\right)x_1 + d^2\left(\sum_{k=0}^{n-1} k^2\right).$$

Equivalently, x_1 is a real root of the following quadratic in x

$$1 = nx^2 + (n-1)nd \cdot x + \frac{1}{6}(n-1)n(2n-1)d^2,$$

which will happen if and only if $n > 1$ and $d > 0$ are such that its discriminant

$$(n-1)^2 n^2 d^2 - \frac{2}{3}(n-1)n^2(2n-1)d^2 + 4n$$

is non-negative and this is equivalent to saying $(n-1)n(n+1)d^2 \leq 12$. Consequently the greatest common difference is

$$d = \sqrt{\frac{12}{(n-1)n(n+1)}}\,,$$

and the first term of the progression is the solution of the quadratic above with this value of d, namely

$$x_1 = -\sqrt{\frac{3(n-1)}{n(n+1)}}.$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. One incomplete solution was received.

3572. [2010 : 397, 399] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\left(\sum_{\text{cyclic}} \frac{ab}{c+ab}\right) + \frac{1}{4}\prod_{\text{cyclic}}\left(\frac{a+\sqrt{ab}}{a+b}\right) \geq 1\,.$$

Composite of similar solutions by Arkady Alt, San Jose, CA, USA; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and Albert Stadler, Herrliberg, Switzerland.

Note first that

$$\sum_{\text{cyclic}} \frac{ab}{c+ab} = \sum_{\text{cyclic}} \frac{ab}{c(a+b+c)+ab} = \sum_{\text{cyclic}} \frac{ab}{(c+a)(c+b)}$$
$$= \frac{1}{(a+b)(b+c)(c+a)} \sum_{\text{cyclic}} ab(a+b)\,.$$

Hence the given inequality is equivalent to

$$4\sum_{\text{cyclic}} ab(a+b) + \prod_{\text{cyclic}} (a+\sqrt{ab}) \geq 4(a+b)(b+c)(c+a),$$

or

$$\prod_{\text{cyclic}} (a+\sqrt{ab}) \geq 8abc\,.$$