Comment by Oliver Geupel, Brühl, NRW, Germany.

The cited article does not contain the promised inequality. Its main result is the weaker inequality

$$PA \cdot QA + PB \cdot QB + PC \cdot QC \geq 4(r_1r_2 + r_2r_3 + r_3r_1), \qquad (1)$$

where r_i denotes the distances from P to the sides of the triangle, and both P and Q are interior points. We prove the following

Lemma. For each point Q outside the triangle there exists a point Q' on the boundary of the triangle such that $QA > QA'$, $QB > QB'$, and $QC > QC'$.

Proof. Drawing rays from the vertices of the triangle orthogonal to the sides partitions the plane outside the triangle into six regions: Three regions S_A, S_B, S_C outwardly on the sides and three regions T_A, T_B, T_C outwardly on the vertices. If Q lies in an "S" region, define Q' to be the orthogonal projection of Q onto the adjacent side of the triangle. If Q lies in an "T" region define Q' to be the adjacent vertex.

Editor's comment. Geupel's lemma implies that inequality (1) holds for all points Q in the plane. The status of the required result for angle bisectors, however, remains in doubt until somebody produces either a correct reference or a valid proof. Of course, Geupel's lemma shows that if the desired inequality holds when Q is an interior point, then it holds for arbitrary Q.

No solutions were received.

3570. [2010 : 397, 399] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let r, r_a, r_b, r_c, and R be, respectively, the inradius, the exradii, and the circumradius of triangle ABC with side lengths a, b, c. Prove that

$$\frac{r_a^2}{a^2 + r_a^2} + \frac{r_b^2}{b^2 + r_b^2} + \frac{r_c^2}{c^2 + r_c^2} \;\geq\; \frac{4R + r}{4R - r}\,.$$

Solution by Kee-Wai Lau, Hong Kong, China.

Applying the Cauchy-Schwarz inequality to vectors

$$\left(\frac{r_a}{\sqrt{a^2 + r_a^2}}, \frac{r_b}{\sqrt{b^2 + r_b^2}}, \frac{r_c}{\sqrt{c^2 + r_c^2}}\right) \text{ and } \left(\sqrt{a^2 + r_a^2}, \sqrt{b^2 + r_b^2}, \sqrt{c^2 + r_c^2}\right)\,,$$

we have

$$\frac{r_a^2}{a^2 + r_a^2} + \frac{r_b^2}{b^2 + r_b^2} + \frac{r_c^2}{c^2 + r_c^2} \geq \frac{(r_a + r_b + r_c)^2}{\left(a^2 + r_a^2\right) + \left(b^2 + r_b^2\right) + \left(c^2 + r_c^2\right)}\,. \qquad (1)$$

Let s be the semiperimeter. We need the following known results.

$$\sum_{\text{cyclic}} r_a = 4R + r, \tag{2}$$

$$\sum_{\text{cyclic}} r_a^2 = (4R + r)^2 - 2s^2, \tag{3}$$

$$\sum_{\text{cyclic}} a^2 = 2\left(s^2 - r^2 - 4Rr\right). \tag{4}$$

Formulas (2) and (3) appear on p.61 (items 99, 103) and formula (4) on p.52 (item 5) [1]. With these formulas, we can get

$$\begin{aligned}\frac{(r_a + r_b + r_c)^2}{\left(a^2 + r_a^2\right) + \left(b^2 + r_b^2\right) + \left(c^2 + r_c^2\right)} &= \frac{(4R + r)^2}{(4R + r)^2 - 2s^2 + 2\left(s^2 - r^2 - 4Rr\right)} \\ &= \frac{(4R + r)^2}{(4R + r)(4R - r)} = \frac{4R + r}{4R - r},\end{aligned}$$

and this with (1) completes the solution.

Also solved by JOE HOWARD, Portales, NM, USA; OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer.

Geupel and the proposer note that the identity $a^2 + b^2 + c^2 + r_a^2 + r_b^2 + r_c^2 = 16R^2 - r^2$, *used in the last step of the featured solution, is interesting on its own.*

References

[1] D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989.

3571. [2010 : 397, 399] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let $n \geq 1$ be an integer. Among all increasing arithmetic progressions x_1, x_2, $\dots$, x_n such that $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, find the progression with the greatest common difference d.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Since the case $n = 1$ is degenerate, let us assume $n > 1$. An arithmetic progression $x_1, x_2, \dots, x_n$ with common difference $d > 0$ has the property that $1 = x_1^2 + x_2^2 + \cdots + x_n^2$ if and only if

$$1 = \sum_{k=0}^{n-1} (x_1 + kd)^2 = nx_1^2 + 2d\left(\sum_{k=0}^{n-1} k\right)x_1 + d^2\left(\sum_{k=0}^{n-1} k^2\right).$$

Equivalently, x_1 is a real root of the following quadratic in x

$$1 = nx^2 + (n-1)nd \cdot x + \frac{1}{6}(n-1)n(2n-1)d^2,$$