3342. [2008 : 240, 242] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let r and R be the inradius and circumradius of $\triangle ABC$, respectively. Prove that

$$2\sum_{\text{cyclic}} \sin\frac{A}{2}\,\sin\frac{B}{2} \;\le\; 1+\frac{r}{R}\,.$$

Similar solutions by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; Oliver Geupel, Brühl, NRW, Germany; and Dung Nguyen Manh, High School of HUS, Hanoi, Vietnam.

It is well known that $\cos A+\cos B+\cos C = 1+\frac{r}{R}$. Thus, it suffices to show that

$$2\sum_{\text{cyclic}} \sin\frac{A}{2}\,\sin\frac{B}{2} \;\le\; \sum_{\text{cyclic}} \cos A\,.$$

For positive x we have $2 \le x+\frac{1}{x}$. Taking $x = \frac{\cos(A/2)}{\cos(B/2)}$ and multiplying by $\sin\frac{A}{2}\sin\frac{B}{2}$, we obtain

$$2\sin\frac{A}{2}\sin\frac{B}{2} \;\le\; \frac{1}{2}\left(\sin A\tan\frac{B}{2}+\sin B\tan\frac{A}{2}\right)\,,$$

hence,

$$2\sum_{\text{cyclic}} \sin\frac{A}{2}\sin\frac{B}{2} \;\le\; \frac{1}{2}\sum_{\text{cyclic}} \tan\frac{A}{2}(\sin B+\sin C)\,.$$

By sum to product formulas, we have

$$\begin{aligned}
&\frac{1}{2}\sum_{\text{cyclic}} \tan\frac{A}{2}(\sin B+\sin C)\\
&\qquad= \sum_{\text{cyclic}} \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}\sin\frac{B+C}{2}\cos\frac{B-C}{2}\\
&\qquad= \sum_{\text{cyclic}} \cos\frac{B+C}{2}\cos\frac{B-C}{2} \;=\; \sum_{\text{cyclic}} \cos A\,,
\end{aligned}$$

as desired.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; SCOTT BROWN, Auburn University, Montgomery, AL, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; and the proposer. There was one incomplete solution submitted.