3341. [2008 : 240, 242] *Proposed by Arkady Alt, San Jose, CA, USA.*

For any triangle ABC with sides of lengths a, b, and c, prove that $\sqrt{3}(R_a + R_b + R_c) \le a + b + c$, where R_a, R_b, and R_c are the distances from the incentre of $\triangle ABC$ to the vertices A, B, and C, respectively.

Solution by George Apostolopoulos, Messolonghi, Greece.

Let s be the semiperimeter of triangle ABC. We have

$$R_a = \frac{s-a}{\cos\frac{A}{2}} = \frac{s-a}{\sqrt{\frac{s(s-a)}{bc}}} = \frac{\sqrt{bc}\sqrt{s-a}}{\sqrt{s}} = \sqrt{bc}\sqrt{1-\frac{a}{s}}\,.$$

Similarly,

$$R_b = \sqrt{ca}\sqrt{1-\frac{b}{s}} \quad \text{and} \quad R_c = \sqrt{ab}\sqrt{1-\frac{c}{s}}\,.$$

Using the Cauchy–Schwarz Inequality, we obtain

$$\begin{aligned} (R_a+R_b+R_c)^2 &= \left(\sqrt{bc}\sqrt{1-\frac{a}{s}} + \sqrt{ca}\sqrt{1-\frac{b}{s}} + \sqrt{ab}\sqrt{1-\frac{c}{s}}\right)^2 \\ &\le (bc+ca+ab)\left(1-\frac{a}{s}+1-\frac{b}{s}+1-\frac{c}{s}\right) \\ &= (ab+bc+ca)\left(3-\frac{2s}{s}\right) = ab+bc+ca\,, \end{aligned}$$

or

$$\sqrt{3}(R_a+R_b+R_c) \le \sqrt{3(ab+bc+ca)}\,.$$

It suffices to show that

$$\sqrt{3(ab+bc+ca)} \le a+b+c\,.$$

The last inequality is equivalent to $a^2+b^2+c^2 \ge ab+bc+ca$, which is well known and easy to prove. This completes the solution.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.