SOLUTIONS

Aucun problème n'est immuable. L'éditeur est toujours heureux d'envisager la publication de nouvelles solutions ou de nouvelles perspectives portant sur des problèmes antérieurs.

3330. [2008 : 171, 174] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let n be a natural number, let r be a real number, and let $a_1, a_2, \ldots, a_n$ be positive real numbers satisfying $\prod_{k=1}^{n} a_k = r^n$; prove that

$$\sum_{k=1}^{n} \frac{1}{(1+a_k)^3} \;\geq\; \frac{n}{(1+r)^3}\,,$$

(a) for $n=2$ if and only if $r \geq \frac{1}{3}$;

(b) for $n=3$ if $r \geq \frac{1}{\sqrt[3]{4}}$;

(c) for $n=4$ if $r \geq \frac{1}{\sqrt[3]{4}}$;

(d) for $n \geq 5$ if and only if $r \geq \sqrt[3]{n} - 1$.

Solution to parts (a)-(c) *by Oliver Geupel, Brühl, NRW, Germany, solution to part* (d) *by the proposer.*

(a) The statement is not correct in the strict sense, because for each $r > 0$ the inequality is satisfied by $a_1 = a_2 = r$ (and similarly for part (d)). We prove instead that for $r > 0$, the inequality

$$\frac{1}{(1+a)^3} + \frac{1}{(1+b)^3} \;\geq\; \frac{2}{(1+r)^3}\,, \tag{1}$$

holds for all positive real numbers a and b satisfying $ab = r^2$ if and only if $r \geq \frac{1}{3}$.

If $ab \geq \frac{1}{9}$, then the inequality (1) follows from the result given in ***CRUX with Mayhem***, problem 3319 (solution at [2009 : 121-122]).

Conversely, suppose that $r < \frac{1}{3}$. Let $f : [0,\infty) \to \mathbb{R}$ be given by

$$f(x) \;=\; \frac{1}{(1+x)^3} + \frac{x^3}{\left(x+r^2\right)^3}\,.$$

We have $f''(r) = \dfrac{6(3r-1)}{r(1+r)^5} < 0$ and f'' is continuous. It follows that there exists an $x_0 > 0$ such that $f(x_0) < f(r) = \dfrac{2}{(1+r)^3}$. We conclude that

$a = x_0$ and $b = \dfrac{r^2}{a}$ violate the inequality (1). This completes the proof of part (a). Equality holds if and only if $a = b = r$.

(b) We prove the result under the less restrictive condition $r \geq 0.47$. Without loss of generality, let $a_3 \leq r$ and put $x = \sqrt{a_1 a_2}$. Then $x \geq r$, and by part (a) we have

$$\frac{1}{(1+a_1)^3} + \frac{1}{(1+a_2)^3} \;\geq\; \frac{2}{(1+x)^3}\,.$$

It therefore suffices to show that

$$\frac{2}{(1+x)^3} + \frac{x^6}{\left(x^2+r^3\right)^3} \;\geq\; \frac{3}{(1+r)^3}\,.$$

Clearing denominators and rearranging terms in this last inequality, we find that it is equivalent to

$$(x-r)^2 \sum_{k=0}^{7} p_k(r) x^k \;\geq\; 0\,,$$

where

$$\begin{aligned}
p_0(r) &= r^7\left(2r^3+6r^2+6r-1\right) \\
p_1(r) &= r^6\left(4r^3+12r^2+3r-2\right) \\
p_2(r) &= r^4\left(6r^4+15r^3+18r^2+15r-3\right) \\
p_3(r) &= r^3\left(5r^4+18r^3+33r^2+5r-6\right) \\
p_4(r) &= r\left(4r^5+21r^4+27r^3+13r^2+9r-3\right) \\
p_5(r) &= r\left(3r^4+15r^3+21r^2+21r-3\right)-6 \\
p_6(r) &= 2r^4+9r^3+15r^2+5r-6 \\
p_7(r) &= r^3+3r^2+3r-2
\end{aligned}$$

It suffices to prove that $p_k(r) > 0$ for $r \geq 0.47$ and $0 \leq k \leq 7$. Using a calculator, we verify that $p_k(0.47) > 0$ for each k. Moreover, the polynomials $p_k(r)$ are increasing functions for real arguments $r \geq 0.47$. This completes the proof. Equality holds if and only if $a_1 = a_2 = a_3 = r$.

(c) We prove the result under the weaker condition $r \geq 0.59$. Without loss of generality, let $a_4 \leq r$ and put $x = \sqrt[3]{a_1 a_2 a_3}$. Then $x \geq r$, and by part (b) we have

$$\frac{1}{(1+a_1)^3} + \frac{1}{(1+a_2)^3} + \frac{1}{(1+a_3)^3} \;\geq\; \frac{3}{(1+x)^3}\,.$$

It therefore suffices to show that

$$\frac{3}{(1+x)^3} + \frac{x^9}{\left(x^3+r^4\right)^3} \;\geq\; \frac{4}{(1+r)^3}\,.$$

Clearing denominators and rearranging terms in this last inequality, we find that it is equivalent to

$$(x-r)^2 \sum_{k=0}^{10} q_k(r)x^k \;\geq\; 0\,,$$

where

$$\begin{aligned}
q_0(r) &= r^{10}\big(3r^3+9r^2+9r-1\big)\\
q_1(r) &= r^9\big(6r^3+18r^2+6r-2\big)\\
q_2(r) &= r^8\big(9r^3+15r^2+3r-3\big)\\
q_3(r) &= r^6\big(8r^4+21r^3+27r^2+23r-3\big)\\
q_4(r) &= r^5\big(7r^4+27r^3+51r^2+13r-6\big)\\
q_5(r) &= r^4\big(6r^4+33r^3+39r^2+3r-9\big)\\
q_6(r) &= r^2\big(5r^5+27r^4+36r^3+20r^2+15r-3\big)\\
q_7(r) &= r\big(4r^5+21r^4+33r^3+37r^2+3r-6\big)\\
q_8(r) &= r\big(3r^4+15r^3+30r^2+18r-9\big)-9\\
q_9(r) &= 2r^4+9r^3+15r^2+3r-9\\
q_{10}(r) &= r^3+3r^2+3r-3
\end{aligned}$$

It suffices to prove that $q_k(r) > 0$ for $r \geq 0.59$ and $0 \leq k \leq 10$. Using a calculator, we verify that $q_k(0.59) > 0$ for each k. Moreover, the polynomials $q_k(r)$ are increasing functions for real arguments $r \geq 0.59$. This completes the proof. Equality holds if and only if $a_1 = a_2 = a_3 = a_4 = r$.

(d) Suppose that r is such that the inequality holds for all $a_1, a_2, \ldots, a_n$ subject to the given constraint. Let x be a positive real number, let $a_i = x$ for $i = 1, 2, \ldots, n-1$, and let $a_n = \dfrac{r^n}{x^{n-1}}$. Then

$$\frac{n-1}{(1+x)^3} \;+\; \frac{x^{3n-3}}{\big(x^{n-1}+r^n\big)^3} \;\geq\; \frac{n}{(1+r)^3}$$

holds for all $x > 0$. Taking the limit as $x \to \infty$ yields $\dfrac{n}{(1+r)^3} \leq 1$, hence $r \geq \sqrt[3]{n}-1$.

Conversely, suppose that $r \geq \sqrt[3]{n}-1$. We will prove by Mathematical Induction that if $n \geq 4$, $r \geq \max\left\{\frac{1}{\sqrt[3]{4}}, \sqrt[3]{n}-1\right\}$, and $a_1, a_2, \ldots, a_n$ satisfy the given constraint, then $\sum\limits_{k=1}^{n} \dfrac{1}{(1+a_k)^3} \geq \dfrac{n}{(1+r)^3}$.

Note that the statement is true for $n = 4$ by part (c).

Now suppose that the statement is true for some $n \geq 4$ and that $r \geq \max\left\{\frac{1}{\sqrt[3]{4}}, \sqrt[3]{n+1}-1\right\} = \sqrt[3]{n+1}-1$, and let $a_1, a_2, \ldots, a_{n+1}$ be positive real numbers such that $a_1a_2\cdots a_{n+1} = r^{n+1}$. By symmetry, we may assume that $a_1 \geq a_2 \geq \cdots \geq a_{n+1}$. Let $x = \sqrt[n]{a_1 a_2 \cdots a_n}$, then $x \geq a_{n+1} = \frac{r^{n+1}}{x^n}$ and $x^{n+1} \geq r^{n+1}$, so that $x \geq r \geq \sqrt[3]{n+1} - 1 > \frac{1}{\sqrt[3]{4}}$. By induction, we have $\sum\limits_{k=1}^{n} \frac{1}{(1+a_k)^3} \geq \frac{n}{(1+x)^3}$, hence

$$\sum_{k=1}^{n} \frac{1}{(1+a_k)^3} \;\geq\; \frac{n}{(1+x)^3} + \frac{x^{3n}}{\left(x^n + r^{n+1}\right)^3}\,.$$

Let $h(x)$ be the function of x on the right side of the above inequality for $x \geq r$. After some (tedious) calculations we find that

$$\begin{aligned} h'(x) &= \frac{3n\left(x^{n+1} - r^{n+1}\right)P(x)}{(1+x)^4\left(x^n + r^{n+1}\right)^4}\,; \\ P(x) &= 6x^{2n}r^{n+1} + 4x^{2n+1}r^{n+1} + 4x^n r^{2n+2} \\ &\qquad + x^{2n+2}r^{n+1} + x^{n+1}r^{2n+2} + r^{3n+3} - x^{3n-1}\,. \end{aligned}$$

Now $P(r) = r^{3n-1}(r+1)^3(3r-1) > 0$, since $r > \frac{1}{\sqrt[3]{4}} > \frac{1}{3}$, and by degree considerations $P(x) \to -\infty$ as $x \to \infty$, hence $P(x)$ has exactly one root $x_0 \in [0,\infty)$. [*Ed.: note that for positive x and positive $C_1, C_2, \ldots, C_n$, the function $\frac{C_n}{x^n} + \frac{C_{n-1}}{x^{n-1}} + \cdots + \frac{C_1}{x} + C_0$ is decreasing, and $\frac{P(x)}{x^{3n-1}}$ is of this form.*] So, $P(x) > 0$ for $x \in [r, x_0)$ and $P(x) < 0$ for $x \in (x_0, \infty)$. Hence, $h(x)$ is increasing on $[r, x_0)$ and decreasing on (x_0, ∞). Thus,

$$\min_{x \in [r, x_0]} h(x) \;=\; h(r) \;=\; \frac{n}{(1+r)^3} + \frac{r^{3n}}{\left(r^n + r^{n+1}\right)^3} \;=\; \frac{n+1}{(1+r)^3}$$

and for any $x \in [x_0, \infty)$ we have

$$h(x) \;>\; \lim_{x \to \infty} h(x) \;=\; 1 \;\geq\; \frac{n+1}{(1+r)^3} \;=\; h(r)\,.$$

Therefore, the minimum value of $h(x)$ on $[r, \infty)$ is $h(r) = \frac{n+1}{(1+r)^3}$, which completes the induction step and the proof.

Also solved by the proposer (parts (a)-(c)). There was one incomplete solution submitted.

*The proposer leaves **Crux** readers with the problem of determining the minimum values of r for which parts (b) and (c) hold.*

3338. [2008 : 239, 242] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

A convex cyclic quadrilateral $ABCD$ has an incircle with centre I. Let P be the intersection of AC and BD. Prove that $AP : CP = AI^2 : CI^2$.