Similar solutions by Walther Janous, Ursulinengymnasium, Innsbruck, Austria and Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain.

The inequality in part (a) is known: see Kiran S. Kedlaya, "Proof of a mixed arithmetic-mean, geometric-mean inequality", *Amer. Math. Monthly*, Vol. 101, No. 4 (1994), pp. 355–357.

To prove part (b), we replace the numbers $a_1, a_2, \dots, a_n$ in part (a) with their reciprocals $\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}$ and for each k we let A'_k and G'_k be the resulting arithmetic and geometric means. Clearly, $G'_k = \frac{1}{G_k}$ and $A'_k = \frac{1}{H_k}$. By part (a) we have

$$\frac{1}{n}\sum_{k=1}^{n} G'_k \le \left(\prod_{k=1}^{n} A'_k\right)^{\frac{1}{n}},$$

therefore,

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{G_k} \le \left(\prod_{k=1}^{n}\frac{1}{H_k}\right)^{\frac{1}{n}} = \left(\prod_{k=1}^{n} H_k\right)^{-\frac{1}{n}}$$

and

$$n\left(\sum_{k=1}^{n}\frac{1}{G_k}\right)^{-1} \ge \left(\prod_{k=1}^{n} H_k\right)^{\frac{1}{n}},$$

as required.

Part (a) also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; and OLIVER GEUPEL, Brühl, NRW, Germany.

Curtis cites the reference in our featured solution and in addition he cites Takashi Matsuda, "An inductive proof of a mixed arithmetic–geometric mean inequality", Amer. Math. Monthly, *Vol. 102, No. 7 (1995), pp. 634–637. He informs us that Kedlaya's proof is combinatorial in nature, while Matsuda's proof uses induction and Lagrange Multipliers.*

3329. [2008 : 171, 173] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let r be a real number, $0 < r \le 1$, and let x, y, and z be positive real numbers such that $xyz = r^3$. Prove that

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+z^2}} \le \frac{3}{\sqrt{1+r^2}}.$$

Solution by Roy Barbara, Lebanese University, Fanar, Lebanon.

First we prove that for $0 < s \le 1$ and $x, y > 0$ such that $xy = s^2$,

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} \le \frac{2}{\sqrt{1+s^2}}. \qquad (1)$$

Let $\alpha = \arctan x$ and $\beta = \arctan y$. Since $\tan\alpha\tan\beta = xy = s^2 \le 1$, we have $\alpha + \beta \le \frac{\pi}{2}$. Thus $\tan\alpha\tan\beta \le \tan^2\left(\frac{\alpha+\beta}{2}\right)$ (see the book by Ivan Niven and Lester H. Lance, *Maxima and Minima Without Calculus*, p. 103). Therefore

$$\begin{aligned}\cos\alpha + \cos\beta &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\\ &\le 2\cos\left(\frac{\alpha+\beta}{2}\right) = \frac{2}{\sqrt{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)}}\\ &\le \frac{2}{\sqrt{1+\tan\alpha\tan\beta}}\,.\end{aligned}$$

This inequality implies (1). For $xyz = r^3$, we have $\min\{xy, yz, zx\} \le 1$. We may assume (by symmetry) that $xy \le 1$. Set $xy = s^2$. By the previous result

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} \le \frac{2}{\sqrt{1+s^2}}$$

and to obtain the given inequality it is enough to prove that if $z > 0$ and $s^2 z = r^3$, then

$$\frac{2}{\sqrt{1+s^2}} + \frac{1}{\sqrt{1+z^2}} \le \frac{3}{\sqrt{1+r^2}}\,. \tag{2}$$

For $z > 0$ let

$$f(z) = \frac{1}{\sqrt{1+z^2}} + \frac{2}{\sqrt{1+\dfrac{r^3}{z}}} = \frac{1}{\sqrt{1+z^2}} + \frac{2\sqrt{z}}{\sqrt{z+r^3}}\,.$$

Direct computation gives $f'(z) = -\frac{z}{(1+z^2)^{3/2}} + \frac{r^3}{\sqrt{z}(z+r^3)^{3/2}}$. Further calculations reveal that $f'(z) = 0$ if and only if $(z-r)\big((1-r^2)z + r\big) = 0$, hence $z = r$ is the only (positive) zero of f'. Since $\lim\limits_{z\to 0^+} f(z) = 1$ and we also have $f(r) = \frac{3}{\sqrt{1+r^2}} \ge \frac{3}{\sqrt{2}} > 2$, it follows that f has an absolute maximum at $z = r$ and the inequality (2) holds.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; ADAM STRZEBONSKI, Wolfram Research Inc., Champaign, IL, USA and STAN WAGON, Macalester College, St. Paul, MN, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There were two incorrect solutions submitted.