Hence, $XD = HD$. Similarly, $YE = HE$ and $ZF = HF$. Let $[ABC]$ denote the area of triangle ABC. We have

$$\begin{aligned}\frac{AX}{AD} + \frac{BY}{BE} + \frac{CZ}{CF} &= \frac{AD+XD}{AD} + \frac{BE+YE}{BE} + \frac{CF+ZF}{CF} \\ &= \frac{AD+HD}{AD} + \frac{BE+HE}{BE} + \frac{CF+HF}{CF} \\ &= 3 + \frac{HD}{AD} + \frac{HE}{BE} + \frac{HF}{CF} \\ &= 3 + \frac{[HBC]}{[ABC]} + \frac{[HAC]}{[ABC]} + \frac{[HAB]}{[ABC]} \\ &= 3 + \frac{[ABC]}{[ABC]} = 4\,.\end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA (2 solutions); KONSTANTINE ZELATOR, University of Toledo, Toledo, OH, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Bataille, Geupel, and Peiró gave solutions using signed distances, for which the featured solution is valid for all triangles, but the statement (as given) is not true for obtuse triangles. For example, if $AB = AC$, then AX is fixed (it is the diameter of the circumcircle), so the ratio $\frac{AX}{AD}$ can be larger than 4; in fact, it can be made arbitrarily large. The correct statement for an obtuse triangle, say, with $\angle A > 90^\circ$, B between E and Y, and C between F and Z would be

$$\frac{AX}{AD} - \frac{BY}{BE} - \frac{CZ}{CF} = 4\,.$$

3319. [2008 : 103, 106] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let m be a natural number, $m \geq 2$, and let r be any real number such that $r \geq 1/m$. If a and b are positive real numbers satisfying $ab = r^2$, prove that

$$\frac{1}{(1+a)^m} + \frac{1}{(1+b)^m} \geq \frac{2}{(1+r)^m}\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

First we show that for all positive real numbers x,

$$\frac{(x+1)^2}{x^{2/(m+1)}} \geq \frac{m+1}{m-1} \cdot \frac{1-r^2}{r^{2/(m+1)}}\,. \qquad (1)$$

To prove (1), consider the function $h(x) = \frac{(x+1)^2}{x^{2/(m+1)}}$ for positive x. From the derivative

$$h'(x) \;=\; \frac{2(x+1)}{x^{(m+3)/(m+1)}}\left(x - \frac{x+1}{m+1}\right),$$

we see that $h'(x) \le 0$ for $x \in \left(0, \frac{1}{m}\right]$ while $h'(x) \ge 0$ for $x \in \left[\frac{1}{m}, \infty\right)$. Therefore, h takes its minimum value at $x = \frac{1}{m}$ and we have

$$h\left(\frac{1}{m}\right) \;=\; \frac{m+1}{m-1}\left(1 - \frac{1}{m^2}\right)m^{2/(m+1)} \;\ge\; \frac{m+1}{m-1} \cdot \frac{1-r^2}{r^{2/(m+1)}},$$

which completes the proof of (1).

Next we claim that

$$k(x) \;=\; r^2 x^{m-1}(1+x)^{m+1} - \left(x + r^2\right)^{m+1}$$

is not positive if $0 < x \le r$ and is not negative if $x \ge r$. For this, consider the function $g(x) = r^{2/(m+1)}x^{(m-1)/(m+1)} - \frac{x+r^2}{x+1}$, for which we have

$$g'(x) \;=\; \frac{m-1}{m+1} \cdot \frac{r^{2/(m+1)}}{x^{2/(m+1)}} - \frac{1-r^2}{(x+1)^2}.$$

Observe that $g(r) = 0$ and, by (1), $g'(x) \ge 0$ for all $x > 0$, which implies our claim regarding $k(x)$.

Finally, we prove the required inequality by writing $x = a$, $b = \frac{r^2}{x}$, and by considering for $x > 0$ the function $f(x) = \frac{1}{(1+x)^m} + \frac{1}{(1+r^2/x)^m}$. We have

$$f'(x) \;=\; \frac{m}{\left((1+x)(x+r^2)\right)^{m+1}}\,k(x).$$

From what we know about $k(x)$ we have that $f'(x) \le 0$ for $0 < x \le r$, and $f'(x) \ge 0$ for $x \ge r$. Therefore, f takes its minimum value at $x = r$, which is $f(r) = \frac{2}{(1+r)^m}$, as desired.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incomplete and three incorrect solutions submitted.

3320. [2008 : 103, 106] *Proposed by Michel Bataille, Rouen, France.*

Let $\triangle ABC$ be right-angled at A and let O be the midpoint of BC. Let M be a point in the plane of $\triangle ABC$, and let M', M'', N, N', and N'' denote the orthocentres of $\triangle MAB$, $\triangle MAC$, $\triangle AM'M''$, $\triangle NAB$, and $\triangle NAC$, respectively. If O is the midpoint of $M'M''$, show that O is also the midpoint of $N'N''$.