Moreover, from $f(a+b) < a$ we obtain $(2a+b)w < 2a(a+b)$. As for the condition $2a(a-b) < (2a-b)w$, it follows from $f(|a-b|) > a$ if $a > b$, and from $(2a-w)(a-b) < aw$ if $a \le b$ (because $2a > w$, the left side is negative). The desired equivalence follows.

(c) Note that $b > |a-w| + \frac{1}{2}h_a$ is equivalent to

$$ab \;>\; a\,|a-w| \;+\; \text{Area}(ABC)\,;$$

that is, to $ab > a\,|a-w| + \left(\frac{a+c}{2}\right) w \sin\frac{B}{2}$. Since $a\,|a-w| < \frac{(2a-w)b}{2}$ (from part (b)), the latter will certainly hold if

$$b \;\ge\; (a+c)\sin\frac{B}{2}\,.$$

This inequality is equivalent to

$$\sin B \;\ge\; (\sin A + \sin C)\sin\frac{B}{2}\,,$$

or to

$$2\cos\frac{B}{2} \;\ge\; 2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)\,,$$

or finally to

$$1 \ge \cos\left(\frac{A-C}{2}\right)\,,$$

which is certainly true. The result follows.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA (part (c) only); and the proposer.

Parts (a) and (b) of our problem appear on page 11 of D.S. Mitrinović et al., Recent Advances in Geometric Inequalities, Kluwer Academic Publishers, *1989 as the first of 40 existence results from a 1952 paper (in Czech) by G. Petrov.*

In addition to his solution, Oxman also addressed the question of constructibility. Exercise 4 on page 142 of Günter Ewald's Geometry: An Introduction *(Wadsworth Publ., 1971) says that in general a triangle cannot be constructed by ruler and compass given the lengths a, b, and w_b, even when that triangle exists. The author suggests that the proof of his claim can be simplified by taking both the given side lengths equal to 1. The formula $f(x) = 1$ from part (a) of the featured solution (with $a = b = 1$, and w^2 chosen to be rational) is a cubic equation with rational coefficients. One simply has to choose a value of w for which the resulting cubic equation has no rational root. The theory of Euclidean constructions then tells us that the positive root, namely c, cannot be constructed by using ruler and compass.*

3300. [2007 : 487, 489] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let a, b, and c be positive real numbers. For any positive integer n define

$$F_n \;=\; \left(\frac{3(a^n+b^n+c^n)}{a+b+c} - \sum_{\text{cyclic}} \frac{b^n+c^n}{b+c}\right)\,.$$

(a) Prove that $F_n \ge 0$ for $n \le 5$.

(b)★ Prove or disprove that $F_n \ge 0$ for $n \ge 6$.

Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

Since $F_1 = 0$, we take $n > 1$. We note that $\left(x^{n-1} - y^{n-1}\right)(x-y) \geq 0$ for all positive x and y, with equality if and only if $x = y$. We have

$$\begin{aligned}
(a+b+c)F_n &= 3\left(a^n + b^n + c^n\right) - (a+b+c)\sum_{\text{cyclic}} \frac{b^n + c^n}{b+c} \\
&= \left(a^n + b^n + c^n\right) - \sum_{\text{cyclic}} \frac{a\left(b^n + c^n\right)}{b+c} \\
&= \sum_{\text{cyclic}} \left[a^n - \frac{a\left(b^n + c^n\right)}{b+c}\right] \\
&= \sum_{\text{cyclic}} \left[\frac{ab\left(a^{n-1} - b^{n-1}\right)}{(b+c)} + \frac{ac\left(a^{n-1} - c^{n-1}\right)}{(b+c)}\right] \\
&= \sum_{\text{cyclic}} \frac{ab\left(a^{n-1} - b^{n-1}\right)(a-b)}{(b+c)(c+a)} \geq 0\,.
\end{aligned}$$

Equality holds if and only if $a = b = c$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; VASILE CÎRTOAJE, University of Ploiesti, Romania; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; NIKOLAOS DERGIADES, Thessaloniki, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PANOS E. TSAOUSSOGLOU, Athens, Greece (part (a) only); STAN WAGON, Macalester College, St. Paul, MN, USA (part (a) only); TITU ZVONARU, Comănești, Romania; and the proposer.

Cîrtoaje mentioned that this problem was posted (together with a solution similar to the one featured above) by Wolfgang Berndt (Spanferkel) on the Mathlinks Forum website `http://www.mathlinks.ro/Forum/viewtopic.php?p=607167` *in August 2006. Barbara, Cîrtoaje, and Dergiades proved the following generalization: If $a_1, a_2, \ldots, a_m$ are positive real numbers, $m \geq 2$, and*

$$F_n = \frac{m(a_1^n + a_2^n + \cdots + a_m^n)}{a_1 + a_2 + \cdots + a_m} - \sum_{\text{cyclic}} \frac{a_2^n + \cdots + a_m^n}{a_2 + \cdots + a_m}\,,$$

then $F_n \geq 0$ for all $n \geq 1$. Alt ultimately proved that if a, b, c, p, and q are positive real numbers and

$$F(p,q) = \frac{3(a^p + b^p + c^p)}{a^q + b^q + c^q} - \sum_{\text{cyclic}} \frac{a^p + b^p}{a^q + b^q}\,,$$

then $(p-q)F(p,q) \geq 0$.