3071. [2005 : 398, 400] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let $k > -1$ be a fixed real number. Let a, b, and c be non-negative real numbers such that $a+b+c=1$ and $ab+bc+ca>0$. Find

$$\min\left\{\frac{(1+ka)(1+kb)(1+kc)}{(1-a)(1-b)(1-c)}\right\}.$$

Solution by Michel Bataille, Rouen, France, modified by the editor.

The required minimum is $\min\left\{\frac{1}{8}(k+3)^3, (k+2)^2\right\}$.

First, we establish the following inequality:

$$4(ab+bc+ca) \;\le\; 1+9abc\,. \tag{1}$$

Using the fact that $a+b+c=1$, we get

$$\begin{aligned} &1-4(ab+bc+ca)+9abc \\ &\qquad = \; (a+b+c)^3-4(a+b+c)(ab+bc+ca)+9abc \\ &\qquad = \; a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b) \;\ge 0\,, \end{aligned}$$

where the last line is Schur's Inequality. This proves (1).

We also claim that

$$ab+bc+ca \;\ge\; 9abc\,. \tag{2}$$

Indeed,

$$\begin{aligned} ab+bc+ca &= (ab+bc+ca)(a+b+c) \\ &= a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+3abc \\ &\ge 6abc+3abc \;=\; 9abc\,, \end{aligned}$$

where the inequality follows by an application of the AM–GM Inequality.

Turning back to the problem, we note that it is not possible for a, b, or c to equal 1. If $a=1$, for example, then $b=c=0$, which means that $ab+bc+ca=0$, a contradiction. Thus, $a, b, c \in [0,1)$. Let

$$\begin{aligned} Q(a,b,c) &= \frac{(1+ka)(1+kb)(1+kc)}{(1-a)(1-b)(1-c)} \\ &= \frac{k^3abc+k^2(ab+bc+ca)+k+1}{ab+bc+ca-abc} \\ &= k^2+(k+1)\frac{k^2abc+1}{ab+bc+ca-abc}\,. \end{aligned}$$

Note that $Q\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)=\frac{1}{8}(k+3)^3$ and $Q\left(0,\frac{1}{2},\frac{1}{2}\right)=(k+2)^2$.

Case 1. $k^2 \le 5$.

We prove that $Q(a,b,c) \ge Q\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$. Since $k+1>0$, a straightforward calculation shows that this inequality is equivalent to

$$k^2(ab+bc+ca-9abc)+27(ab+bc+ca-abc) \;\le\; 8\,. \tag{3}$$

The term involving k^2 is non-negative, in view of (2). Since $k^2 \le 5$, the left side of (3) is at most $8\big(4(ab+bc+ca) - 9abc\big)$ and (3) follows from (1). Thus, $Q\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{8}(k+3)^3$ is the minimum value of Q.

Case 2. $k^2 \ge 5$.

We prove that $Q(a,b,c) \ge Q\left(0, \frac{1}{2}, \frac{1}{2}\right)$. Since $k+1 > 0$, we find that this inequality is equivalent to

$$1 + 4\big(abc - (ab+bc+ca)\big) + k^2abc \;\ge\; 0\,.$$

This holds by (1) because $k^2 \ge 5$. Thus, $Q\left(0, \frac{1}{2}, \frac{1}{2}\right) = (k+2)^2$ is the minimum value of Q.

Noticing that

$$\tfrac{1}{8}(k+3)^3 - (k+2)^2 \;=\; \tfrac{1}{8}(k+1)(k^2-5)\,,$$

we see that $\frac{1}{8}(k+3)^3 \ge (k+2)^2$ if $k^2 \ge 5$ and $(k+2)^2 \ge \frac{1}{8}(k+3)^3$ if $k^2 \le 5$. The announced result follows.

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