Problem 11790
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Given a triangle with semiperimeter s, inradius r, and medians of length m_a, m_b, and m_c, prove that

$$m_a + m_b + m_c \leq 2s - 3(2\sqrt{3} - 3)r.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let us consider the triangle ABC and let A' and B' be the symmetric points of A and B with respect to the midpoints of BC and CA respectively. Then, by Ptolemy's inequality applied to the quadrilateral $ABA'B'$, we have that

$$4m_a m_b = (2m_a)(2m_b) \leq ab + (2c)c = ab + 2c^2.$$

In a similar way, $4m_b m_c \leq bc + 2a^2$, $4m_c m_a \leq ca + 2b^2$ and by adding all together we find that

$$2(m_a m_b + m_b m_c + m_c m_a) \leq a^2 + b^2 + c^2 + \frac{1}{2}(ab + bc + ca).$$

Moreover, $4m_a^2 = 2b^2 + 2c^2 - a^2$ and similar formulas imply

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2).$$

Hence

$$(m_a + m_b + m_c)^2 \leq \frac{7}{4}(a^2 + b^2 + c^2) + \frac{1}{2}(ab + bc + ac).$$

Since

$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} \quad \text{and} \quad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

it is easy to verify that

$$a^2 + b^2 + c^2 = 2s^2 - 8Rr - 2r^2 \quad \text{and} \quad ab + bc + ac = s^2 + 4Rr + r^2.$$

Therefore

$$(m_a + m_b + m_c)^2 \leq 4s^2 - 12Rr - 3r^2,$$

and it suffices to prove that

$$4s^2 - 12Rr - 3r^2 \leq (2s - 3(2\sqrt{3} - 3)r)^2$$

that is

$$s \leq \frac{R + (16 - 9\sqrt{3})r}{2\sqrt{3} - 3}.$$

Finally, by Blundon's inequality, $s \leq 2R + (3\sqrt{3} - 4)r$. So we still have to show that

$$2R + (3\sqrt{3} - 4)r \leq \frac{R + (16 - 9\sqrt{3})r}{2\sqrt{3} - 3}$$

which is equivalent to Euler's inequality $R \geq 2r$. □

Remark. For Blundon's inequality see *Inequalities associated with the triangle*, Canad. Math. Bull., 8 (1965), 615-626 and Problem E1935, The Amer. Math. Monthly, 73 (1966), 1122.