A Minimization with Sum and Product

11718 [2013, 570]. *Proposed by Arkady Alt, San Jose, CA.* Given positive real numbers $a_1, \dots, a_n$ with $n \geq 2$, minimize $\sum_{i=1}^n x_i$ subject to the conditions that $x_1, \dots, x_n$ are positive and that $\prod_{i=1}^n x_i = \sum_{i=1}^n a_i x_i$.

Solution by Ronald E. Prather, Oakland, CA. Let

$$S = \left\{ (x_1, \dots, x_n) : x_i > 0, \prod_{i=1}^n x_i = \sum_{i=1}^n a_i x_i \right\}.$$

First consider what happens when a point approaches the boundary of S. Writing the constraint as

$$1 = \sum_{i=1}^n \frac{a_i}{\prod_{j \neq i} x_j},$$

we see that each product $\prod_{j \neq i} x_i$ is bounded away from zero. If any x_k tends to zero, another one (in fact at least two) must tend to infinity, and then $\sum_{i=1}^n x_i$ tends to infinity. Therefore, $\sum_{i=1}^n x_i$ achieves a minimum value in the interior of S. We will find it using Lagrange multipliers.

Writing μ for the reciprocal of the usual Lagrange multiplier, we get $n + 1$ equations

$$\prod_{j \neq i} x_j = a_i + \mu \quad \text{for } 1 \leq i \leq n, \quad \text{and} \prod_{i=1}^n x_n = \sum_{i=1}^n a_i x_i$$

in the $n + 1$ unknowns $x_1, \dots, x_n$ and μ. Substituting the first n equations in the last yields an nth degree polynomial equation for μ, namely $f(\mu) = 0$, where

$$f(x) = \sum_{i=1}^n \frac{a_i}{a_i + x}.$$

Now f is continuous and monotonically decreasing, with $f(0) = n > 1$ and $f(\infty) = 0 < 1$, so there is a unique positive solution μ to the equation $f(\mu) = 1$. Multiplying the ith equation by x_i and summing over i, we get $n \prod_{j=1}^n x_j = \sum_{i=1}^n a_i x_i + \mu \sum_{i=1}^n x_i = \prod_{j=1}^n x_j + \mu \sum_{i=1}^n x_i$, so

$$\sum_{i=1}^n x_i = \frac{n-1}{\mu} \prod_{j=1}^n x_j.$$

Multiplying the first n equations yields $\prod_{i=1}^n x_i^{n-1} = \prod_{i=1}^n (a_i + \mu)$. Thus the minimum value of $\sum_{i=1}^n x_i$ is

$$\frac{n-1}{\mu} \left(\prod_{j=1}^n (a_j + \mu) \right)^{1/(n-1)}.$$

Editorial comment. The proposer notes that the problem is related to Problem 4 of the 2001 Vietnam Team Selection Test.

Also solved by R. Bagby, R. Boukharfane (Canada), P. Bracken, M. Dincă (Romania), N. Grivaux (France), O. Kouba (Syria), J. Martínez (Spain), N. C. Singer, R. Stong, T. Viteam (Chile), GCHQ Problem Solving Group (U. K.), and the proposer.