H-840 Proposed by Arkady Alt, **San Jose** , **California**, **USA**.

Prove that $(n-1)(n+1)(2nF_{n+1}-(n+6)F_n)$ is divisible by 150 for any $n \in \mathbb{N}$.

(F_n is $n-th$ Fibonacci number, $n \in \mathbb{N}$)

Solution.

Let $s_n := \dfrac{(n-1)(n+1)(2nF_{n+1}-(n+6)F_n)}{150}, n \in \mathbb{N} \cup \{0\}$. Then $s_0 = 0, s_1 = 0$.

Note that $s_{n+1} = \dfrac{n(n+2)(2(n+1)F_{n+2}-(n+7)F_{n+1})}{150} =$

$\dfrac{n(n+2)(2(n+1)(F_{n+1}+F_n)-(n+7)F_{n+1})}{150} = \dfrac{n(n+2)((n-5)F_{n+1}+2(n+1)F_n)}{150}$

and

$s_{n-1} = \dfrac{(n-2)n(2(n-1)F_n-(n+5)F_{n-1})}{150} = \dfrac{(n-2)n(2(n-1)F_n-(n+5)(F_{n+1}-F_n))}{150} =$

$\dfrac{(n-2)n(3(n+1)F_n-(n+5)F_{n+1})}{150}$. Then

$s_{n+1}-s_n-s_{n-1} = \dfrac{n(n+2)((n-5)F_{n+1}+2(n+1)F_n)}{150} - \dfrac{(n-1)(n+1)(2nF_{n+1}-(n+6)F_n)}{150} -$

$\dfrac{(n-2)n(3(n+1)F_n-(n+5)F_{n+1})}{150} = \dfrac{(5n^2+3n-2)F_n-6nF_{n+1}}{50}$.

Let $h_n := \dfrac{(5n^2+3n-2)F_n-6nF_{n+1}}{50}, n \in \mathbb{N} \cup \{0\}$. Then $h_0 = 0, h_1 = 0$.

Note that $h_{n+1} = \dfrac{\left(5(n+1)^2+3(n+1)-2\right)F_{n+1}-6(n+1)F_{n+2}}{50} =$

$\dfrac{(5n^2+13n+6)F_{n+1}-6(n+1)(F_{n+1}+F_n)}{50} = \dfrac{n(5n+7)F_{n+1}-6(n+1)F_n}{50}$.

Then

$h_{n+2} = \dfrac{(n+1)(5(n+1)+7)F_{n+2}-6(n+2)F_{n+1}}{50} = \dfrac{(5n^2+17n+12)(F_{n+1}+F_n)-6(n+2)F_{n+1}}{50} =$

$\dfrac{(5n^2+17n+12)F_n+n(5n+11)F_{n+1}}{50}$ and, therefore,

$h_{n+2}-h_{n+1}-h_n = \dfrac{(5n^2+17n+12)F_n+n(5n+11)F_{n+1}}{50} - \dfrac{n(5n+7)F_{n+1}-6(n+1)F_n}{50} -$

$\dfrac{(5n^2+3n-2)F_n-6nF_{n+1}}{50} = \dfrac{nF_{n+1}+2(n+1)F_n}{5}$.

Let $g_n := \dfrac{nF_{n+1}+2(n+1)F_n}{5},\ n \in \mathbb{N} \cup \{0\}$. Then $g_0 = 0, g_1 = \dfrac{F_2+4F_1}{5} = 1$ and

we have $g_{n+1} = \dfrac{(n+1)F_{n+2}+2(n+2)F_{n+1}}{5} = \dfrac{(n+1)(F_{n+1}+F_n)+2(n+2)F_{n+1}}{5} =$

$\dfrac{(n+1)F_n+(3n+5)F_{n+1}}{5}$. Also, $g_{n-1} = \dfrac{(n-1)F_n+2nF_{n-1}}{5} = \dfrac{(n-1)F_n+2n(F_{n+1}-F_n)}{5} =$

$\dfrac{2nF_{n+1}-(n+1)F_n}{5}$.

Hence,

$g_{n+1}-g_n-g_{n-1} = \dfrac{(n+1)F_n+(3n+5)F_{n+1}}{5} - \dfrac{nF_{n+1}+2(n+1)F_n}{5} - \dfrac{2nF_{n+1}-(n+1)F_n}{5} = F_{n+1}$.

Since g_n is integer for any $n \in \mathbb{N} \cup \{0\}$ (By Math Induction because $g_0 = 0, g_1 = 1$ and $g_{n+1} = g_n + g_{n-1} + F_{n+1}, n \in \mathbb{N}$) and $h_{n+2} = h_{n+1}+h_n+g_n, n \in \mathbb{N} \cup \{0\}$, where $h_0 = h_1 = 0$, then for any $n \in \mathbb{N} \cup \{0\}$ h_n is integer as well. Thus, $s_{n+1} = s_n + s_{n-1} + h_n, n \in \mathbb{N} \cup \{0\}$, where $s_0 = 0, s_1 = 0$ and h_n is integer for any $n \in \mathbb{N} \cup \{0\}$.

Hence, again by Math Induction, we can conclude that s_n is integer for any $n \in \mathbb{N} \cup \{0\}$.