$$Re\left(\sum_{k=1}^{n} z_k w_k\right) \le \left|\sum_{k=1}^{n} z_k w_k\right| \le \sum_{k=1}^{n} |z_k w_k|\,,$$

and the inequality amounts to show that

$$2\sqrt{\frac{3}{20}\frac{3n^2+6n+1}{n^2+3n+2}} \ge 1 \iff n \le -\frac{7}{4},\ n \ge 1.$$

This completes the proof.

Solution 4 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

Let $z_k = x_k + iy_k$ and $w_k = a_k + ib_k$, for $0 \le k \le n$. We can assume that $x_k, y_k, a_k, b_k \ge 0$, because we can increase the left hand side of the statement of the problem by using absolute values.

We wish to prove the inequality:

$$\sum_{k=1}^{n}(a_k x_k - b_k y_k) \le \frac{3}{(n+1)(n+2)}\sum_{k=1}^{n}\left(x_k^2 + y_k^2\right) + \frac{3n^2+6n+1}{20}\sum_{k=1}^{n}\left(a_k^2 + b_k^2\right).$$

Because of symmetry, we need only show that:

$$a_k x_k \le \frac{3}{(n+1)(n+2)}x_k^2 + \frac{3n^2+6n+1}{20}a_k^2.$$

Considering this as a quadratic inequality for the variable x_k, we see that the discriminant is negative.

$$\Delta = a_k^2 - 4\frac{3}{(n+1)(n+2)}\frac{3n^2+6n+1}{20}a_k^2 = a_k^2\left(\frac{-4n^2+3n+7}{5(n+1)(n+2)}\right) < 0.$$

Hence, the problem is solved.

Also solved by Bruno Salgueiro Fanego,Viveiro, Spain; Ed Gray, Highland Beach, FL, and the proposer.

- **5342:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a, b, c, \alpha > 0$, be real numbers. Study the convergence of the integral

$$I(a,b,c,\alpha) = \int_1^{\infty}\left(a^{1/x} - \frac{b^{1/x}+c^{1/x}}{2}\right)^{\alpha} dx.$$

The problem is about studying the conditions which the four parameters, a, b, c, and α, should verify such that the improper integral would converge.

Solution 1 by Arkady Alt, San Jose, CA

Case 1. If $a = b = c$, then for any nonzero x, $a^{\frac{1}{x}} - \dfrac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2} = 0$, so $I(a, b, c, \alpha) = 0$ for any real $\alpha > 0$.

Case 2. Suppose α isn't an integer. Then $a^{\frac{1}{x}} - \dfrac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2}$ must be nonnegative for any x and in particular, it must be positive for $x = 1$, that is $a \geq \dfrac{b+c}{2}$.

Since $\begin{cases} 2a = b + c \\ b = c \end{cases} \iff a = b = c$ then, to avoid the trivial case 1, we will consider a, b, c such that

$a > \dfrac{b+c}{2}$ or $\begin{cases} 2a = b + c \\ b \neq c. \end{cases}$

Then, by the AM-PM inequality, for $x > 1$ we have

$$\frac{b+c}{2} > \left(\frac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2} \right)^x \iff \left(\frac{b+c}{2} \right)^{\frac{1}{x}} > \frac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2},$$

and we obtain $a^{\frac{1}{x}} > \dfrac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2}$ for any $x > 1$ and that the integral is defined.

For any real $p > 0$ we have $\lim\limits_{t \to 0} \dfrac{p^t - 1}{t} = \ln p$. So, $\lim\limits_{x \to \infty} x \left(a^{\frac{1}{x}} - \dfrac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2} \right) =$

$$\lim_{x \to \infty} x \left(a^{\frac{1}{x}} - 1 \right) - \frac{1}{2} \left(\lim_{x \to \infty} x \left(b^{\frac{1}{x}} - 1 \right) + \lim_{x \to \infty} x \left(c^{\frac{1}{x}} - 1 \right) \right) = \ln a - \frac{\ln b + \ln c}{2} = \ln \frac{a}{\sqrt{bc}} > 0,$$

because $a > \sqrt{bc}$ if $b \neq c$ or if $a > \dfrac{b+c}{2}$.

Therefore, $\lim\limits_{x \to \infty} \dfrac{\left(a^{\frac{1}{x}} - \dfrac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2} \right)^\alpha}{\dfrac{1}{x^\alpha}} = \ln^\alpha \dfrac{a}{\sqrt{bc}} > 0$, and by the Limit Comparison Test, $I(a, b, c, \alpha)$ converges iff $\dfrac{1}{x^\alpha}$ converges; that is, $I(a, b, c, \alpha)$ converges if $\alpha > 1$ and diverges if $\alpha \in (0, 1]$.

Case 3. Let α be a positive integer. Then the expression $\left(a^{\frac{1}{x}} - \dfrac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2} \right)^\alpha$ is defined for any positive a, b, c and since

$$\lim_{x \to \infty} \left(a^{\frac{1}{x}} - \frac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2} \right)^\alpha = \ln^\alpha \frac{a}{\sqrt{bc}} > 0$$

is the limit of $I(a, b, c, \alpha)$ for $a > \sqrt{bc}$ and when $\alpha > 1$. So the situation of $a = \sqrt{bc}$ must be analyzed.

Then $\left(a^{\frac{1}{x}} - \frac{b^{\frac{1}{x}}+c^{\frac{1}{x}}}{2}\right)^{\alpha} = \frac{(-1)^{\alpha}\left(b^{\frac{1}{2x}} - c^{\frac{1}{2x}}\right)^{2\alpha}}{2^{\alpha}}.$

Assume, without loss of generality, $b > c$. Since $\lim_{x\to\infty} x\left(b^{\frac{1}{2x}} - a^{\frac{1}{2x}}\right) = \frac{1}{2}\ln\frac{b}{c} > 0,$

then $\lim_{x\to\infty} \frac{\left(b^{\frac{1}{2x}} - a^{\frac{1}{2x}}\right)^{2\alpha}}{\frac{1}{x^{2\alpha}}} = \left(\frac{1}{2}\ln\frac{b}{c}\right)^{2\alpha} > 0,$ and by the Limit Comparison Test

$I(a,b,c,\alpha)$ is convergent iff $\frac{1}{x^{2\alpha}}$ convergent, that is $I(a,b,c,\alpha)$ convergent if $\alpha > 1/2$ and divergent if $\alpha \in (0, 1/2]$.

In summary,

- If $a = b = c$ then $I(a,b,c,\alpha) = 0$ is convergent for any real α;
- If $\alpha \in \Re_+/N$ and $a > \frac{b+c}{2}$ or $\begin{cases} 2a = b+c \\ b \neq c \end{cases}$ then $I(a,b,c,\alpha)$ is convergent for $\alpha > 1$ and divergent for $\alpha \in (0,1]$;
- If $\alpha \in \Re_+/N$ and $a > \sqrt{bc}$ then $I(a,b,c,\alpha)$ is convergent for $\alpha > 1$ and divergent for $\alpha \in (0,1]$;
- If $\alpha \in N$ and $a = \sqrt{bc}$ then $I(a,b,c,\alpha)$ is convergent for $\alpha > 1/2$ and divergent for $\alpha \in (0,1/2]$.

Solution 2 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

To have the integral well defined, a necessary condition is $2a \geq b + c$.

The convergence occurs in one of the following cases:

1) if $a = b = c$ we have convergence for any value of α

2) if $\alpha > 1$ we have convergence regardless the values of a, b, c

3) if $1/2 < \alpha \leq 1$ and $a = \sqrt{bc}$ we have convergence.

Proof
If α is irrational or it is a rational p/q reduced to the lowest terms with q even, we must impose

$$2a^{1/x} - b^{1/x} - c^{1/x} \geq 0$$

but this doesn't seem to me easy to prove. A necessary condition is $2a \geq b + c$ corresponding to $x = 1$.

If $a = b = c$ the integrand is identically zero and then the integral converges regardless the value of α.

From now on, $a \neq b$ or $b \neq c$ or $a \neq c$.
We have $a^{1/x} = e^{\frac{\ln a}{x}} = 1 + \frac{\ln a}{x} + \frac{\ln^2 a}{2x^2} + \frac{\ln^3 a}{6x^3} + O(x^{-4})$ whence