Let $f(x) = \dfrac{-1+(n-1)x^2+(n+1)x^4}{x(x^2+1)}$. One can easy observe that

$$
\begin{aligned}
f'(x) &= \frac{1+(n+2)x^2+(2n+4)x^4+(n+1)x^6}{x^2(1+x^2)^2}\\
f''(x) &= -\frac{2(1+3x^2+2x^6)}{x^3(1+x^2)^3}
\end{aligned}
$$

It is obvious that $f'(x) > 0$ and $f''(x) < 0$ for any real positive number x, which implies that the function $f(x)$ is an increasing and concave function in the real positive domain. Applying Jensen's inequality we have

$$\sum_{cycl} \frac{-1+(n-1)x^2+(n+1)x^4}{x(x^2+1)} = \sum_{cycl} f(x) \le 3f\left(\frac{1}{3}\sum_{cycl} x\right)$$

Doing easy manipulations, one can easy observe that

$$\alpha = \sum_{cycl} \frac{(n+1)x^3+nx}{x^2+1} = \sum_{cycl} nx + \sum_{cycl} \frac{x^3}{x^2+1} > n\sum_{cycl} x$$

Finally, using the above results we have

$$
\begin{aligned}
\sum_{cycl} \frac{1}{x} &= \alpha - \sum_{cycl} \frac{-1+(n-1)x^2+(n+1)x^4}{x(x^2+1)}\\
&\ge \alpha - 3f\left(\frac{1}{3}\sum_{cycl} x\right)\\
&> \alpha - 3f\left(\frac{\frac{\alpha}{n}}{3}\right)\\
&= \alpha - 3f\left(\frac{\alpha}{3n}\right)\\
&= \frac{9n}{\alpha} - \frac{\alpha}{n} + \frac{9n\alpha}{9n^2+\alpha^2}
\end{aligned}
$$

and this is the end of the proof.

- **5275:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Find all real solutions to the following system of equations

$$
\left.
\begin{array}{c}
\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+x_1}}}}_{n} + \underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2+x_1}}}}_{n} = x_2\sqrt{2},\\
\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+x_2}}}}_{n} + \underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2+x_2}}}}_{n} = x_3\sqrt{2},\\
\ldots\ldots\ldots\ldots\ldots\ldots\\
\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+x_{n-1}}}}}_{n} + \underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2+x_{n-1}}}}}_{n} = x_n\sqrt{2},\\
\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+x_n}}}}_{n} + \underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2+x_n}}}}_{n} = x_1\sqrt{2},
\end{array}
\right\}
$$

where $n \geq 2$.

Solution by Arkady Alt, San Jose, CA

Let $h(x) := \sqrt{2+x}$. Then $h(x)$ is a function defined on $[-2,\infty)$ with range $[0,\infty)$.

Since $h : [-2,\infty) \longrightarrow [0,\infty)$ then for any $n \in N$ we can define recursively $n-$iterated function $h_n : [-2,\infty) \longrightarrow [0,\infty)$, namely $h_1 = h$ and $h_{n+1} = h \circ h_n, n \geq 1$.

Let $f(x) := \dfrac{h_n(x) + \sqrt{2 - h_{n-1}(x)}}{\sqrt{2}}$ for $x \in [-2,\infty)$ such that $h_{n-1}(x) \leq 2$.

Since $h_{n-1}(x) \leq 2 \iff h_{n-1}^2(x) \leq 4 \iff h_{n-2}(x) \leq 2 \iff .... \iff h_1(x) \leq 2 \iff x \leq 2$

then $Dom(f) = [-2, 2]$. Moreover, applying inequality $\dfrac{a+b}{\sqrt{2}} \leq \sqrt{a^2+b^2}$ to $a = h_n(x)$ and $b = \sqrt{2 - h_{n-1}(x)}$ we obtain $f(x) \leq 2$ and since by definition $f(x) \geq 0$ for $x \in Dom(f)$
then $range(f) \subset [0, 2]$.

Using f we can rewrite original system as follow:

(1) $$\begin{cases} x_{k+1} = f(x_k), k = 1, 2, ..., n-1 \\ x_1 = f(x_n) \end{cases}.$$

Since $x_k \in [0, 2]$, $k = 1, 2, ..., n$ then by setting $t_k := \cos^{-1}\left(\dfrac{x_k}{2}\right), k = 1, 2, ..., n$ we obtain $t_k \in \left[0, \dfrac{\pi}{2}\right]$, $x_k = 2\cos t_k$, $k = 1, 2, ..., n$.

Noting that $h(2\cos t) = 2\cos t2$ for $t \in \left[0, \dfrac{\pi}{2}\right]$ by Math. Induction we obtain $h_k(2\cos t) = 2\cos\dfrac{t}{2^k}, k = 1, 2, ..., ...$and, therefore, $f(2\cos t) =$

$$\frac{1}{\sqrt{2}}\left(2\cos\frac{t}{2^n} + \sqrt{2 - 2\cos\frac{t}{2^{n-1}}}\right) = 2\left(\frac{1}{\sqrt{2}}\cos\frac{t}{2^n} + \frac{1}{\sqrt{2}}\sin\frac{t}{2^n}\right) = 2\cos\left(\frac{\pi}{4} - t2^n\right).$$

Since $\dfrac{\pi}{4} - \dfrac{t}{2^n} \in \left[0, \dfrac{\pi}{2}\right]$ for $t \in \left[0, \dfrac{\pi}{2}\right]$ then $\dfrac{\pi}{4} - \dfrac{t_k}{2^n} \in \left[0, \dfrac{\pi}{2}\right]$ as well as $t_k \in \left[0, \dfrac{\pi}{2}\right]$ for any $k = 1, 2, ..., n$ and, therefore, **(1)**

$$\iff \begin{cases} 2\cos t_{k+1} = 2\cos\left(\dfrac{\pi}{4} - \dfrac{t_k}{2^n}\right), k = 1, 2, ..., n-1 \\ 2\cos t_1 = 2\cos\left(\dfrac{\pi}{4} - \dfrac{t_n}{2^n}\right) \end{cases} \iff$$

(2) $$\begin{cases} t_{k+1} = \dfrac{\pi}{4} - \dfrac{t_k}{2^n}, k = 1, 2, ..., n-1 \\ t_1 = \dfrac{\pi}{4} - t_n 2^n \end{cases}.$$

Lemma:

Let a, b be real numbers such that $|a| \neq 1$. Then system of equations

$$\begin{cases} t_{k+1} = b + at_k, k = 1, 2, ..., n-1 \\ t_1 = b + at_n \end{cases}$$

have only solution $t_1 = t_2 = ... = t_n = \dfrac{b}{1-a}$.

Proof: Noting that $\dfrac{b}{1-a}=b+a\cdot\dfrac{b}{1-a}$ and denoting $c:=\dfrac{b}{1-a}$ we obtain
$t_{k+1}=b+at_k \iff t_{k+1}-c=a\left(t_k-c\right), k=1,2,...n-1$
and $t_1=b+at_n \iff t_1-c=a\left(t_n-c\right)$. Since $t_k-c, k=1,2,...$
is geometric sequence we have $t_k-c=a^{k-1}\left(t_1-c\right), k=1,2,...n-1$ and therefore,

$t_1-c=a\cdot a^{n-1}\left(t_1-c\right) \iff t_1-c=a^n\left(t_1-c\right) \iff \left(t_1-c\right)\left(1-a^n\right)=0 \iff t_1=c.$

That yield $t_k-c=a^{k-1}\left(t_1-c\right)=0 \iff t_k=c, k=2,...,n.$

Thus, $t_1=t_2=...=t_n=c=\dfrac{b}{1-a}$.

Applying the Lemma with $a=-\dfrac{1}{2^n}$ and $b=\dfrac{\pi}{4}$ we obtain the only solution of **(2),**

$t_1=t_2=...=t_n=\dfrac{2^{n-2}\pi}{2^n+1}$ and then $x_1=x_2=...=x_n=2\cos\left(\dfrac{2^{n-2}\pi}{2^n+1}\right)$ is the only solution of original system.

Also solved by Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; and the proposer.

- **5276:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

(a) Let $a\in(0,1]$ be a real number. Calculate

$$\int_0^1 a^{\left\lfloor \frac{1}{x}\right\rfloor}dx,$$

where $\lfloor x\rfloor$ denotes the floor of x.

(b) Calculate

$$\int_0^1 2^{-\left\lfloor \frac{1}{x}\right\rfloor}dx.$$

Solution 1 by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

(a) Using the substitution $1/x=y$, the integral becomes $I=\displaystyle\int_1^\infty a^{\lfloor y\rfloor}/y^2dy$. For any positive integer k and $y\in[k,k+1)$ we have $\lfloor y\rfloor=k$. Then

$$\begin{aligned}
I &= \sum_{k=1}^{\infty}\int_k^{k+1}a^k/y^2dy=\sum_{k=1}^{\infty}a^k\left(\frac{1}{k}-\frac{1}{k+1}\right)\\
&= \sum_{k=1}^{\infty}\frac{a^k}{k}-\sum_{k=1}^{\infty}\frac{a^k}{k+1}(\text{ since both series are absolutely convergent})\\
&= -\ln(1-a)+\frac{\ln(1-a)+a}{a}.
\end{aligned}$$

Since $\displaystyle\sum_{k=1}^{\infty}a^k=\frac{1}{1-a}$, and $\dfrac{a^k}{k}=\displaystyle\int_0^a x^{k-1}dx$ for $k\geq 1$.