Calculate

$$\int_0^1 x \ln\left(\sqrt{1+x}-\sqrt{1-x}\right)\ln\left(\sqrt{1+x}+\sqrt{1-x}\right)dx.$$

Solutions

- **5259:** *Proposed by Kenneth Korbin, New York, NY*

Find $a, b,$ and c such that with $a < b < c$, $\begin{cases} ab+bc+ca &= -2 \\ a^2b^2+b^2c^2+c^2a^2 &= \ \ 6 \\ a^3b^3+b^3c^3+c^3a^3 &= -11. \end{cases}$

Solution 1 by Arkady Alt, San Jose, CA

Let $s = a+b+c,\ p = ab+bc+ca$, and $q = abc$. Then a, b, c are the roots of the equation $x^3 - sx^2 + px - q = 0$. Since,

$$\begin{aligned} 6 &= a^2b^2+b^2c^2+c^2a^2 = p^2 - 2sq = 4 - 2sq \ \text{ and} \\ -11 &= a^3b^3+b^3c^3+c^3a^3 = 3q^2 + p^3 - 3spq = 3q^2 - 8 + 6sq, \ \text{then} \\ sq &= -1 \text{ and } q^2 = 1 \iff q = 1 \text{ or } q = -1. \end{aligned}$$

Thus we obtain $(s, p, q) = (-1, -2, 1), (1, -2, -1)$ and, respectively, the two equations

$$x^3 + x^2 - 2x - 1 = 0 \quad \text{and} \quad x^3 - x^2 - 2x + 1 = 0.$$

Since,

$$\begin{aligned} (-x)^3 + (-x)^2 - 2(-x) - 1 = 0 &\iff x^3 - x^2 - 2x + 1 = 0, \ \text{and} \\ x^3 + x^2 - 2x - 1 = 0 &\iff x = 1.2470,\ -0.44504,\ -1.8019, \end{aligned}$$

we see that,

$$(a, b, c) = (-1.8019,\ -0.44504,\ 1.2470),\ (-1.2470,\ 0.44504,\ 1.8019).$$

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

As in problem 5135, let $x = ab,\ y = bc$ and $z = ca$, so that
$x+y+z = -2,\ x^2+y^2+z^2 = 6$, and $x^3+y^3+z^3 = -1$. We have

$$abc\,(a+b+c) = xy+yz+zx = \frac{(x+y+z)^2 - x^2 - y^2 - z^2}{2} = \frac{(-2)^3 - 6}{2} = -1, \ \text{and}$$

$$(abc)^3 = xyz = \frac{x^3+y^3+z^3 - (x+y+z)\left(x^2+y^2+z^2-xy-yz-zx\right)}{3} = \frac{-11+2(6+1)}{3} = 1.$$