But

$$2010 \cdot 5^6 = 31406250.$$

So the last six digits are 406250.

Solution 4 by Kee-Wai Lau, Hong Kong, China

We show that $(2010)\left(5^{2014}\right) = \cdots 406250$.

It is easy to check that

$$(2010)\left(5^{2014}\right) = 406250 + \left(2^4\right)\left(5^6\right)\left(5^{2011}-1\right) + (2)\left(5^7\right)\left(5^{2008}-1\right).$$

Hence to prove our result, we need only show that $5^{2011}-1$ is a multiple of 4 and $5^{2008}-1$ is a multiple of 32.

In fact,

$$\begin{aligned} 5^{2011} - 1 \equiv 1^{2011} - 1 &\equiv 0 \pmod 4), \text{ and} \\ 5^{2008} - 1 = 390625^{251} - 1 &\equiv 1^{251} - 1 \equiv 0 \pmod{32}, \end{aligned}$$

and this completes the solution.

Also solved by Daniel Lopez Aguayo, UNAM Morelia, Mexico; Brian D. Beasley, Clinton, SC; Pat Costello, Richmond, KY; Bruno Salgueiro Fanego, Viveiro Spain; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5197:** *Proposed by Pedro H. O. Pantoja, UFRN, Brazil*

Let x, y, z be positive real numbers such that $x^2+y^2+z^2=4$. Prove that,

$$\frac{1}{6-x^2} + \frac{1}{6-y^2} + \frac{1}{6-z^2} \leq \frac{1}{xyz}.$$

Solution 1 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

The inequality is evidently

$$\sum_{\text{cyc}} \frac{1}{2+x^2+y^2} \leq \frac{1}{xyz}.$$

$a^2+1 \geq 2|a|$ yields

$$\sum_{\text{cyc}} \frac{1}{2+x^2+y^2} \leq \sum_{\text{cyc}} \frac{1}{2x+2y} \leq \frac{1}{xyz}$$

and $(\sqrt{x}-\sqrt{y})^2 \geq 0$ yields

$$\sum_{\text{cyc}} \frac{1}{2x+2y} \leq \sum_{\text{cyc}} \frac{1}{4\sqrt{xy}} \leq \frac{1}{xyz} \quad \Longleftrightarrow \quad \sum_{\text{cyc}} z\sqrt{xy} \leq 4$$

which is implied by

$$\sum_{\text{cyc}} z\frac{1}{2}(x+y) \le 4 \iff xy+yz+zx \le 4.$$

But this follows by the well known $xy+yz+zx \le x^2+y^2+z^2$, thus concluding the proof.

Soluiton 2 by David E. Manes, Oneonta, NY

Let $L = \dfrac{1}{6-x^2} + \dfrac{1}{6-y^2} + \dfrac{1}{6-z^2}$. Since $x^2+y^2+z^2 = 4$, it follows that

$$6-x^2 = 2+y^2+z^2,\ 6-y^2 = 2+x^2+z^2,\ 6-z^2 = 2+x^2+y^2.$$

Therefore,

$$L = \frac{1}{6-x^2} + \frac{1}{6-y^2} + \frac{1}{6-z^2} = \frac{1}{2+y^2+z^2} + \frac{1}{2+x^2+z^2} + \frac{1}{2+x^2+y^2}.$$

Using the Arithmetic Mean-Geometric Mean Inequality twice, one obtains

$$\begin{aligned} L &= \frac{1}{2+(y^2+z^2)} + \frac{1}{2+(x^2+z^2)} + \frac{1}{2+(x^2+y^2)} \\ &\le \frac{1}{2+(2yz)} + \frac{1}{2+(2xz)} + \frac{1}{2+(2xy)} \\ &= \frac{1}{2}\left(\frac{1}{1+yz} + \frac{1}{1+xz} + \frac{1}{1+xy}\right) \\ &\le \frac{1}{2}\left(\frac{1}{2\sqrt{yz}} + \frac{1}{2\sqrt{xz}} + \frac{1}{2\sqrt{xy}}\right) \\ &= \frac{1}{4}\left(\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{\sqrt{xyz}}\right). \end{aligned}$$

As a result, to show that $L \le \dfrac{1}{xyz}$ it suffices to show that

$$\begin{aligned} \frac{1}{4}\left(\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{\sqrt{xyz}}\right) &\le \frac{1}{xyz}, \text{ if and only if} \\ \frac{1}{4}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right) &\le \frac{1}{\sqrt{xyz}}, \text{ if and only if} \\ \frac{1}{4}\left(x\sqrt{yz}+y\sqrt{xz}+z\sqrt{xy}\right) &\le 1. \end{aligned}$$

However, the Cauchy-Schwarz inequality, and the inequality $xy+yz+zx \le x^2+y^2+z^2$ (*which also follows from the C-S inequality; editor's comment*) imply that

$$\frac{1}{4}\left(x\sqrt{yz}+y\sqrt{xz}+z\sqrt{xy}\right) \le \frac{1}{4}\sqrt{x^2+y^2+z^2}\sqrt{yz+xz+xy}$$

$$\leq \frac{1}{4}\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2+z^2} = 1.$$

Accordingly, if $x, y, z > 0$, and $x^2+y^2+z^2 = 4$, then

$$\frac{1}{6-x^2}+\frac{1}{6-y^2}+\frac{1}{6-z^2} \leq \frac{1}{xyz}.$$

Solution 3 by Arkady Alt, San Jose, CA

Let $a := \frac{x^2}{4}$ $b := \frac{y^2}{4}$, $c := \frac{z^2}{4}$ then inequality becomes

$$\frac{1}{6-4a}+\frac{1}{6-4b}+\frac{1}{6-4c} \leq \frac{1}{8\sqrt{abc}},$$

where $a+b+c = 1$.

Let $E = E(a,b,c) := \sqrt{abc}\sum_{cyc}\frac{1}{3-2a}$, $p := ab+bc+ca, q := abc$.

Since $\sum_{cyc}(3-2b)(3-2c) = \sum_{cyc}(9-6(b+c)+4bc) = 15+4p$,

$(3-2a)(3-2b)(3-2c) = 9+12p-8q$ then $E = \frac{(15+4p)\sqrt{q}}{9+12p-8q}$.

Since $q \leq \frac{p^2}{3}^*$ and E is increasing in q then

$$\frac{E}{\sqrt{3}} \leq \frac{(15+4p)p}{27+36p-8p^2}$$

$$\leq \frac{\left(15+4\cdot\frac{1}{3}\right)\cdot\frac{1}{3}}{27+36\cdot\frac{1}{3}-8\cdot\frac{1}{9}} = \frac{1}{7}$$

because $\frac{(15+4p)p}{27+36p-8p^2}$ is increasing in positive p and

$p \leq \frac{1}{3} \iff ab+bc+ca \leq \frac{(a+b+c)^2}{3}$.

Thus,

$$E \leq \frac{\sqrt{3}}{7} \iff 4E \leq \frac{4\sqrt{3}}{7}$$

$$\iff 8\sqrt{abc}\sum_{cyc}\frac{1}{6-4a} \leq \frac{4\sqrt{3}}{7}$$

$$\iff xyz\sum_{cyc}\frac{1}{6-x^2} \leq \frac{4\sqrt{3}}{7}$$

$$\iff \sum_{cyc} \frac{1}{6-x^2} \leq \frac{4\sqrt{3}}{7xyz}.$$

$$\frac{1}{6-x^2} + \frac{1}{6-y^2} + \frac{1}{6-z^2} \leq \frac{4\sqrt{3}}{7xyz}.$$

Remark: Since $\frac{4\sqrt{3}}{7} < 1$ we have $\frac{1}{6-x^2} + \frac{1}{6-y^2} + \frac{1}{6-z^2} \leq \frac{4\sqrt{3}}{7xyz} < \frac{1}{xyz}$.

So, the inequality in the formulation of problem could have been stated with the stronger statement

$$\frac{1}{6-x^2} + \frac{1}{6-y^2} + \frac{1}{6-z^2} \leq \frac{4\sqrt{3}}{7xyz},\text{instead of with the weaker one of}$$

$$\frac{1}{6-x^2} + \frac{1}{6-y^2} + \frac{1}{6-z^2} \leq \frac{1}{xyz}.$$

* *Editor's comment:* The inequality $q \leq \frac{p^2}{3}$ is equivalent to $3abc(a+b+c) \leq (ab+bc+ca)^2$ which is equivalent to $abc(a+b+c) \leq a^2b^2+b^2c^2+c^2a^2$ which is implied by adding up $a^2bc \leq 0.5a^2(b^2+c^2)$ and its cyclic variants.

Also solved by Kee-Wai Lau*, Hong Kong, China; Ecole Suppa, Teramo, Italy; Albert Stadler*, Herrliberg, Switzerland; Titu Zvonaru, Comănesti, Romania jointly with Neculai Stanciu, Buzău, Romania, and the proposer. (* Observed, specifically stated and proved the stricter inequality.)

- **5198:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let m, n be positive integers. Calculate,

$$\sum_{k=1}^{2n} \prod_{i=0}^{m} \left(\lfloor \frac{k+1}{2} \rfloor + a + i \right)^{-1},$$

where a is a nonnegative number and $\lfloor x \rfloor$ represents the greatest integer less than or equal to x.

Solution 1 by Arkady Alt, San Jose, CA

$$\sum_{k=1}^{2n} \prod_{i=0}^{m} \left(\left\lfloor \frac{k+1}{2} \right\rfloor + a + i \right)^{-1}$$

$$= \sum_{k=1}^{n} \prod_{i=0}^{m} \left(\left\lfloor \frac{2k-1+1}{2} \right\rfloor + a + i \right)^{-1} + \sum_{k=1}^{n} \prod_{i=0}^{m} \left(\left\lfloor \frac{2k+1}{2} \right\rfloor + a + i \right)^{-1}$$

$$= 2\sum_{k=1}^{n} \prod_{i=0}^{m} (k+a+i)^{-1}$$