

Let $f : [0, 1] \rightarrow (0, \infty)$ be a continuous function. Find the value of

$$\lim_{n \rightarrow \infty} \left(\frac{\sqrt[n]{f(\frac{1}{n})} + \sqrt[n]{f(\frac{2}{n})} + \cdots + \sqrt[n]{f(\frac{n}{n})}}{n} \right)^n.$$

Solutions

- **5164:** *Proposed by Kenneth Korbin, New York, NY*

A triangle has integer length sides (a, b, c) such that $a - b = b - c$. Find the dimensions of the triangle if the inradius $r = \sqrt{13}$.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

If a, b and c are the side lengths of the triangle then the inradius r is given by the formula

$$r = \frac{1}{2} \sqrt{\frac{(b+c-a)(c+a-b)(a+b-c)}{a+b+c}}. \quad (\text{see, e.g., } \text{http://mathworld.wolfram.com/Inradius.html}).$$

By assumption, $c = 2b - a$. So

$$\sqrt{13} = \frac{1}{2} \sqrt{\frac{(3b-2a)(2a-b)}{3}}, \text{ or equivalently}$$

$$(3b-2a)(2a-b) = 156.$$

Obviously b is even. (If b were odd, then both $3b - 2a$ and $2a - b$ are odd, and therefore their product would be odd, which is not true.) So $b = 2b'$ and this gives the equation

$$(3b' - a)(a - b') = 39.$$

Note that $39 = xy$ is the product of two integers. So,

$$(x, y) \in \{(1, 39), (3, 13), (13, 3), (39, 1), (-1, -39), (-3, -13), (-13, -3), (-39, -1)\}.$$

If $3b' - a = x$ and $a - b' = y$, then

$$b' = \frac{x+y}{2}, \text{ and}$$

$$a = \frac{x+3y}{2}.$$

We find $(a, b, c) \in \{(59, 40, 21), (21, 16, 11), (11, 16, 21), (21, 40, 59)\}$, and we easily verify that each triplet satisfies the triangle inequality.

Solution 2 by Arkady Alt, San Jose, CA

Let F and s be the area and semiperimeter. Since $a + c = 2b$ then $s = \frac{a+b+c}{2} = \frac{3b}{2}$,

and using $F = \sqrt{s(s-a)(s-b)(s-c)} = sr$ we obtain

$$\begin{aligned}
(s-a)(s-b)(s-c) = sr^2 &\iff \left(\frac{3b}{2} - a\right) \left(\frac{3b}{2} - b\right) \left(\frac{3b}{2} - c\right) = 13 \cdot \frac{3b}{2} \\
&\iff \left(\frac{3b}{2} - a\right) \left(\frac{3b}{2} - c\right) = 39 \\
&\iff \left(\frac{9b^2}{4} - (a+c)\frac{3b}{2} + ac\right) = 39 \iff \left(\frac{9b^2}{4} - 2b \cdot \frac{3b}{2} + ac\right) = 39 \\
&\iff 4ac - 3b^2 = 12 \cdot 13.
\end{aligned}$$

Thus we have

$$\begin{cases} a+c=2b \\ 4ac-3b^2=156 \end{cases} \implies \begin{cases} 4a(2b-a)-3b^2=156 \\ c=2b-a \end{cases} \quad \text{if, and only if,}$$

$$\begin{cases} 4a(2b-a)-3b^2=156 \\ c=2b-a \end{cases} \iff \begin{cases} 8ab-a^2-3b^2=156 \\ c=2b-a. \end{cases}$$

Since $8ab - a^2 - 3b^2 = (3b - 2a)(2a - b)$ and

$$\begin{cases} a < s \\ b < s \\ c < s \end{cases} \iff \begin{cases} 2a < 3b \\ c < s \end{cases} \iff \begin{cases} 2a < 3b \\ 2(2b-a) < 3b \end{cases} \iff b < 2a < 3b$$

then the problem is equivalent to the system

$$(1) \quad \begin{cases} (3b-2a)(2a-b) = 156 \\ b < 2a < 3b. \end{cases}$$

Since $3b - 2a \equiv 2a - b \pmod{2}$ and $156 = 2^2 \cdot 3 \cdot 13 = 2 \cdot 78 = 6 \cdot 26$ then (1) in positive integers is equivalent to

$$\begin{cases} 3b-2a=k \\ 2a-b=m \end{cases} \iff \begin{cases} 2b=k+m \\ 4a=k+3m \end{cases} \iff \begin{cases} a = \frac{k+3m}{4} \\ b = \frac{k+m}{2} \end{cases},$$

where $(k, m) \in \{(2, 78), (78, 2), (6, 26), (26, 6)\}$.

Noting that the inequality $b < 2a < 3b \iff \frac{k+m}{2} < \frac{k+3m}{2} < \frac{3(k+m)}{2}$ holds for any positive k, m we finally obtain

$$(a, b) \in \{(59, 40), (21, 40), (21, 16), (11, 16)\}.$$

Thus, $(a, b, c) \in \{(59, 40, 21), (21, 40, 59), (21, 16, 11), (11, 16, 21)\}$ are all solutions of the problem.

Comment by David Stone and John Hawkins, Statesboro, GA. In their featured solutions to SSM 5146 (May 2011 issue) both Kee-Wai Lau and Brian Beasley found all integral triangles with in-radius $\sqrt{13}$. Note that the condition $a - b = b - c$ is equivalent to $b = (a + c)/2$. That is, irrespective of how one might label or order the sides, the side b must be the “middle-length” side, the average of the other two sides.

Also solved by Brain D. Beasley, Clinton, SC; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie (jointly), San Angelo TX; Bruno Salgueiro Fanego, Viveiro, Spain; Tania Moreno García, University of Holguín (UHO), Holguín, Cuba jointly with José Pablo Suárez Rivero, University of Las Palmas de Gran Canaria (ULPGC), Spain; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriptel, Germany; Sugie Lee, John Patton, and Matthew Fox (jointly; students at Taylor University), Upland, IN; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA; Jim Wilson, Athens, GA, and the proposer.

- **5165:** *Proposed by Thomas Moore, Bridgewater, MA*

“Dedicated to Dr. Thomas Koshy, friend, colleague and fellow Fibonacci enthusiast.”

Let $\sigma(n)$ denote the sum of all the different divisors of the positive integer n . Then n is perfect, deficient, or abundant according as $\sigma(n) = 2n$, $\sigma(n) < 2n$, or $\sigma(n) > 2n$. For example, 1 and all primes are deficient; 6 is perfect, and 12 is abundant. Find infinitely many integers that are not the product of two deficient numbers.

Solution 1 by Kee-Wai Lau, Hong Kong, China

Let $p_1 = 2, p_2 = 3, p_3 = 5, \dots$ be the sequence of primes. We show that for any positive integer n , the integer $\prod_{k=1}^{n+10} p_k$ is not the product of two deficient numbers.

Suppose, on the contrary, that $\prod_{k=1}^{n+10} p_k = ab$, where both a and b are deficient numbers.

Clearly a and b are relatively prime and so

$$4 \left(\prod_{k=1}^{n+10} p_i \right) = 4ab > \sigma(a)\sigma(b) = \sigma(ab) = \sigma \left(\prod_{k=1}^{n+10} p_k \right) = \prod_{k=1}^{n+10} (1 + p_k).$$

Hence,

$$4 > \prod_{k=1}^{n+10} \left(1 + \frac{1}{p_k} \right) \geq \prod_{k=1}^{11} \left(1 + \frac{1}{p_k} \right) = \frac{3822059520}{955049953} = 4.0019\dots,$$

which is a contradiction. This completes the solution.

Solution 2 by Stephen Chou, Talbot Knighton, and Tom Peller (students at Taylor University), Upland, IN

All negative numbers have the same numerical divisors as their positive counterparts;