

W6. (Solution by the proposer.) From

$$\begin{aligned}
 r &= 4R \cdot \sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2} \Rightarrow \frac{r}{R} = \\
 &= 4 \cdot \sqrt{\frac{(p-b)(p-c)(p-c)(p-a)(p-a)(p-b)}{a^2b^2c^2}} = \\
 &= 4 \cdot \frac{(p-a)(p-b)(p-c)}{abc} = 4 \cdot \frac{(b+c-a)(c+a-b)(b+c-a)}{8abc} \Rightarrow \\
 &\Rightarrow \frac{r}{R} = \frac{(b+c-a)(c+a-b)(b+c-a)}{2abc}
 \end{aligned}$$

With the help of Ravi substitutions, $\alpha = y + z$, $b = x + z$, $c = x + y$, demonstrated inequality becomes:

$$\begin{aligned}
 &\frac{y+z}{2x+y+z} + \frac{x+z}{2y+x+z} + \frac{x+y}{2z+x+y} + \frac{8xyz}{2(x+y)(y+z)(z+x)} \leq 2 \Leftrightarrow \\
 &\Leftrightarrow \frac{y+z}{2x+y+z} + \frac{x+z}{2y+x+z} + \frac{x+y}{2z+x+y} + \frac{4xyz}{(x+y)(y+z)(z+x)} \leq 2
 \end{aligned}$$

Then,

$$\frac{x^2}{(x+y)(x+z)} + \frac{y^2}{(y+z)(y+x)} \geq \frac{(x+y)^2}{(x+y)(x+y+2z)} = \frac{x+y}{x+y+2z}$$

Analogous,

$$\frac{y^2}{(y+z)(y+x)} + \frac{z^2}{(z+x)(z+y)} \geq \frac{y+z}{2x+y+z}$$

and

$$\frac{z^2}{(z+x)(z+y)} + \frac{x^2}{(x+y)(x+z)} \geq \frac{z+x}{2y+x+z}$$

Summing up,

$$\begin{aligned}
& 2 \left(\frac{x^2}{(x+y)(x+z)} + \frac{y^2}{(y+z)(y+x)} + \frac{z^2}{(z+x)(z+y)} \right) \geq \\
& \geq \frac{x+y}{x+y+2z} + \frac{y+z}{2x+y+z} + \frac{z+x}{2y+x+z} \Rightarrow \\
& \Rightarrow \frac{2(x^2z + x^2y + y^2z + z^2x + z^2y)}{(x+y)(y+z)(z+x)} \geq \frac{x+y}{x+y+2z} + \frac{y+z}{2x+y+z} + \frac{z+x}{2y+x+z} \Leftrightarrow \\
& \Leftrightarrow \frac{2[(x+y)(y+z)(z+x) - 2xyz]}{(x+y)(y+z)(z+x)} \geq \frac{x+y}{x+y+2z} + \frac{y+z}{2x+y+z} + \frac{z+x}{2y+x+z} \Rightarrow \\
& \Rightarrow 2 - \frac{4xyz}{(x+y)(y+z)(z+x)} \geq \frac{x+y}{x+y+2z} + \frac{y+z}{2x+y+z} + \frac{z+x}{2y+x+z} \Rightarrow \\
& \Rightarrow \frac{y+z}{2x+y+z} + \frac{x+z}{2y+x+z} + \frac{x+y}{2z+x+y} + \frac{4xyz}{(x+y)(y+z)(z+x)} \leq 2,
\end{aligned}$$

which ends demonstration.

Second solution. Let $x := s - a, y := s - b, z := s - c$ where s is semiperimeter of the triangle. Then $x, y, z > 0$. Assume (due homogeneity of original inequality) $s = 1$ and let $p := xy + yz + zx, q := xyz$. Then $x + y + z = 1, a = 1 - x, b = 1 - y, c = 1 - z, abc = p - q, r = sr = [ABC] = \sqrt{q}$,

$$\begin{aligned}
R &= \frac{abc}{4[ABC]} = \frac{p-q}{4\sqrt{q}}, \sum_{cyc} \frac{a}{b+c} = \sum_{cyc} \frac{1-x}{1+x} = \sum_{cyc} \frac{2-(1+x)}{1+x} = 2 \sum_{cyc} \frac{1}{1+x} - 3 = \\
&= \frac{2 \sum_{cyc} (1+y)(1+z)}{(1+x)(1+y)(1+z)} - 3 = \frac{2 \sum_{cyc} (2-x+yz)}{(1+x)(1+y)(1+z)} - 3 = \frac{2(5+p)}{2+p+q} - 3
\end{aligned}$$

and original inequality becomes

$$\frac{2(5+p)}{2+p+q} - 3 + \frac{4q}{p-q} \leq 2 \iff \frac{2(5+p)}{2+p+q} + \frac{4q}{p-q} \leq 5 \iff$$

$$\Longleftrightarrow 2(5+p)(p-q) + 4q(2+p+q) \leq 5(2+p+q)(p-q) \Longleftrightarrow$$

$$\Longleftrightarrow 9q^2 + 2q(p+4) - 3p^2 \leq 0$$

Since

$$3p = 3(xy + yz + zx) \leq (x + y + z)^2 = 1$$

and

$$p^2 = (xy + yz + zx)^2 \geq 3xyz(x + y + z) = 3q$$

then

$$9q^2 + 2q(p+4) - 3p^2 \leq 9\left(\frac{p^2}{3}\right)^2 + 2 \cdot \frac{p^2}{3}(p+4) - 3p^2 = -\frac{p^2(p+1)(1-3p)}{3} \leq 0.$$

Arkady Alt

Third solution. After calculations we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{r}{R} \leq 2 \Leftrightarrow$$

$$\Leftrightarrow \sum aR(a+b)(c+a) + r \prod (a+b) \leq 2R \prod (a+b) \Leftrightarrow$$

$$\Leftrightarrow R \sum a^3 + (r-R) \sum a \sum ab + (2R-r)abc \leq 0$$

With usual notations, we get

$$a+b+c = 2s, abc = aRrs, \sum a^3 = 2s(s^2 - 3r^2 - 6Rr), \sum ab = a^2 + r^2 + 4Rr$$

$$R \sum a^3 + (r-R) \sum a \sum ab + (2R-r)abc \leq 0 \Leftrightarrow$$

$$\Leftrightarrow 2sR(s^2 - 3r^2 - 6Rr) + 2s(r-R)(s^2 + r^2 + 4Rr) + 4Rrs(2R-r) \leq 0 \Leftrightarrow$$

$$\Leftrightarrow 2sr(s^2 + r^2 - 2Rr - 6R^2) \leq 0$$

Applying Gerretsen's inequality, we obtain

$$s^2 \leq 4R^2 + 4Rr + 3r^2$$

$$R^2 - Rr - 2r^2 \geq 0 \Rightarrow (R - 2r)(R + r) \geq 0,$$

that is true.

Nicușor Zlota

W7. (Solution by the proposer.) Note $\beta\gamma = a$, $\alpha\gamma = b$, $\alpha\beta = c$. For the first part of inequality:

$$\sin \alpha + \sin \beta + \sin \gamma - \sin \alpha \cdot \sin \beta \cdot \sin \gamma \geq \sin^3 \alpha + \sin^3 \beta + \sin^3 \gamma \Leftrightarrow$$

$$\Leftrightarrow \frac{a+b+c}{2R} - \frac{abc}{8R^3} \geq \frac{a^3+b^3+c^3}{8R^3} \Leftrightarrow 4R^2(a+b+c) - abc \geq a^3+b^3+c^3 \Leftrightarrow$$

$$\sum a^3 = 2p(p^2 - 3r^2 - 6rR) \Leftrightarrow_{abc=4Rrp} 4R^2 \cdot 2p - 4Rrp \geq 2p(p^2 - 3r^2 - 6rR) \Leftrightarrow$$

$$\Leftrightarrow 8R^2 - 4Rr \geq 2p^2 - 6r^2 - 12rR \Leftrightarrow 2p^2 \leq 8rR + 8R^2 - 4Rr \Leftrightarrow$$

$$\Leftrightarrow p^2 \leq 4R^2 + 4rR + 3r^2,$$

which represents inequality Gerretsen.

a). Suppose $\max(\alpha, \beta, \gamma) < \frac{\pi}{2}$. For starters we demonstrate the following lemma:

Lemma. If M is a point in plan of triangle ABC , then the inequality

$$AB \cdot BC \cdot CA \leq MA^2 \cdot BC + MB^2 \cdot AC + MC^2 \cdot BC$$

Proof. In the complex plane either x, y, z affixes points A, B, C , and M the landmark XOY . Using equality

$$x^2(y-z) + y^2(z-x) + z^2(x-y) = (x-y)(y-z)(z-x)$$

we can write

$$|(x-y)(y-z)(z-x)| = |x^2(y-z) + y^2(z-x) + z^2(x-y)| \leq$$

$$\leq |x|^2 |y - z| + |y|^2 |z - x| + |z|^2 |x - y|,$$

from where

$$AB \cdot BC \cdot CA \leq MA^2 \cdot BC + MB^2 \cdot AC + MC^2 \cdot BC$$

Now, let H orthocentre of triangle $\alpha\beta\gamma$ ($H \in \text{Int}(\triangle ABC)$), and D, E, F the means sides $\beta\gamma, \alpha\gamma$ respectively $\alpha\beta$. According to the Lemma, used for the triangle DEF and $H \in (DEF)$, we have:

$$DE \cdot EF \cdot FD \leq HD^2 \cdot EF + HE^2 \cdot DF + HF^2 \cdot DE \quad (1)$$

Clear $EF = \frac{a}{2}, DE = \frac{c}{2}, DF = \frac{b}{2}$ in triangle $H\beta\gamma$, applying median theorem, we have

$$HD^2 = R^2 [2(\cos^2 \beta + \cos^2 \gamma) - \sin^2 \alpha]$$

Considering that in a triangle we have identity

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 - 2 \cos \alpha \cdot \cos \beta \cdot \cos \gamma,$$

obtain that

$$HD^2 = R^2 [\sin^2 \alpha - 4 \cos \alpha \cdot \cos \beta \cdot \cos \gamma]$$

Substituting in (1) and analogous, we get:

$$\frac{abc}{8} \leq R^2 [\sin^2 \alpha - 4 \cos \alpha \cdot \cos \beta \cdot \cos \gamma] \cdot \frac{a}{2} + R^2 [\sin^2 \beta - 4 \cos \alpha \cdot \cos \beta \cdot \cos \gamma] \cdot \frac{b}{2} +$$

$$+ R^2 [\sin^2 \gamma - 4 \cos \alpha \cdot \cos \beta \cdot \cos \gamma] \cdot \frac{c}{2} \Rightarrow R^3 \sin \alpha \sin \beta \sin \gamma \leq$$

$$\leq R^3 (\sin^3 \alpha + \sin^3 \beta + \sin^3 \gamma - 4 \cos \alpha \cdot \cos \beta \cdot \cos \gamma (\sin \alpha + \sin \beta + \sin \gamma)),$$

from where

$$\sin^3 \alpha + \sin^3 \beta + \sin^3 \gamma \geq \sin \alpha \sin \beta \sin \gamma + 4 \cos \alpha \cdot \cos \beta \cdot \cos \gamma (\sin \alpha + \sin \beta + \sin \gamma)$$

b). If $\max(\alpha, \beta, \gamma) \geq \frac{\pi}{2}$, then $\cos \alpha \cdot \cos \beta \cdot \cos \gamma \leq 0$. Thus

$$\sin^3 \alpha + \sin^3 \beta + \sin^3 \gamma \geq 3 \sqrt[3]{\sin^3 \alpha \cdot \sin^3 \beta \cdot \sin^3 \gamma} = 3 \sin \alpha \sin \beta \sin \gamma >$$

$$> \sin \alpha \sin \beta \sin \gamma + 4 \cos \alpha \cdot \cos \beta \cdot \cos \gamma \cdot (\sin \alpha + \sin \beta + \sin \gamma)$$

Second solution. Let a, b, c be sidelengths of the triangle and R, r, s be circumradius, inradius and semiperimeter, respectively. Then

$$\begin{aligned} & \sum_{cyc} \sin A - \prod_{cyc} \sin A \geq \\ & \geq \sum_{cyc} \sin^3 A \geq \prod_{cyc} \sin A + 4 \prod_{cyc} \cos A \cdot \sum_{cyc} \sin A \iff 8R^3 \\ & 8R^3 \sum_{cyc} \sin A - 8R^3 \prod_{cyc} \sin A \geq \\ & \geq 8R^3 \sum_{cyc} \sin^3 A \geq 8R^3 \prod_{cyc} \sin A + 4 \prod_{cyc} \cos A \cdot 8R^3 \sum_{cyc} \sin A \iff \\ & 8R^2 s - abc \geq a^3 + b^3 + c^3 \geq abc + 32R^2 s \prod_{cyc} \cos A \iff \\ & 8R^2 s - abc \geq a^3 + b^3 + c^3 \geq abc + 32R^2 s \cdot \frac{s^2 - (2R + r)^2}{4R^2} \iff \\ & 8R^2 s - abc \geq a^3 + b^3 + c^3 \geq abc + 8s \left(s^2 - (2R + r)^2 \right) \iff \\ & 8R^2 s - 4Rrs \geq 2s \left(s^2 - 6Rr - 3r^2 \right) \geq 4Rrs + 8s \left(s^2 - (2R + r)^2 \right) \iff \\ & 4R^2 - 2Rr \geq s^2 - 6Rr - 3r^2 \geq 2Rr + 4 \left(s^2 - (2R + r)^2 \right) \quad (1) \\ \text{I. } 4R^2 - 2Rr \geq s^2 - 6Rr - 3r^2 & \iff s^2 \leq 4R^2 + 4Rr + 3r^2 \text{ (Geretsen's} \\ \text{Inequality).} \\ \text{II. } s^2 - 6Rr - 3r^2 \geq 2Rr + 4 \left(s^2 - (2R + r)^2 \right) & \iff 3s^2 \leq 16R^2 + 8Rr + r^2. \\ \text{Since } s^2 \leq 4R^2 + 4Rr + 3r^2 \text{ and } R \geq 2r \text{ (Euler's Inequality) then} \\ & 16R^2 + 10Rr + r^2 - 3s^2 = \end{aligned}$$

$$16R^2 + 8Rr + r^2 - 3(4R^2 + 4Rr + 3r^2) + 3(4R^2 + 4Rr + 3r^2 - s^2) =$$

$$4(R - 2r)(R + r) + 3(4R^2 + 4Rr + 3r^2 - s^2) \geq 0.$$

Arkady Alt

Third solution. With usual notations, we have

1).

$$\frac{a+b+c}{2R} - \frac{abc}{8R^3} \geq \frac{a^3+b^3+c^3}{8R^3} \Leftrightarrow 4R^2s - 4Rrs \geq 2s(s^2 - 3r^2 - 6Rr) \Leftrightarrow$$

$$\Leftrightarrow 2s(4R^2 - 2Rr - s^2 + 3r^2 + 6Rr) \geq 0 \Rightarrow s^2 \leq 4R^2 + 3r^2 + 4Rr$$

which is Gerretsen's inequality

2).

$$\frac{a^3+b^3+c^3}{8R^3} \geq \frac{abc}{8R^3} + 4 \frac{s^2 - (2R+r)^2}{4R^2} \cdot \frac{a+b+c}{2R} \Leftrightarrow 2s(-3s^2 + 16R^2 + 8Rr + r^2) \geq$$

Applying inequality Gerretsen, obtain

$$R^2 - Rr - 2r^2 \geq 0 \Rightarrow (R - 2r)(R + r) \geq 0,$$

that is true.

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W8. (Solution by the proposer.) If $f(x) = 0, \forall x \in [0, 1]$, the conclusion is clear. Suppose there $x_0 \in (0, 1]$ such that

$$f(x_0) \neq 0 \Rightarrow \int_0^1 f(x) dx > 0$$

We note $t = \frac{\int_0^1 g(x) dx}{\int_0^1 f(x) dx} \geq 0$. Then

$$\int_0^1 tf(x) dx = \int_0^1 g(x) dx \Rightarrow \int_0^1 (tf(x) - g(x)) dx = 0$$

Then, from the mean value theorem there $c \in (0, 1)$ such that $tf(c) = g(c)$.
Now, using the

$$\int_0^1 tf(x) dx = \int_0^1 g(x) dx$$

we can write

$$\int_0^c (g(x) - tf(x)) dx = \int_c^1 (tf(x) - g(x)) dx \quad (1)$$

Furthure, how f it's convex, $f(0) = 0$, $g(x)$ it's concave, $g(0) = 0$ and $t \geq 0$, result $tf(x) - g(x)$ it is a convex function, and the function $\frac{tf(x)-g(x)}{x}$ is an increasing function on $(0, 1]$. Where we: for

$$0 < x \leq c \Rightarrow \frac{tf(x) - g(x)}{x} \leq \frac{tf(c) - g(c)}{c} = 0 \Rightarrow g(x) \geq tf(x),$$

and for

$$c \leq x \leq 1 \Rightarrow \frac{tf(x) - g(x)}{x} \geq \frac{tf(c) - g(c)}{c} = 0 \Rightarrow tf(x) \geq g(x).$$

Now,

$$\begin{aligned} \int_0^c h(x) (g(x) - tf(x)) & \stackrel{\substack{h(x) \leq h(c) \\ p(x) - tf(x) \geq 0}}{\leq} \int_0^c h(c) (g(x) - tf(x)) = \\ & = h(c) \int_0^c (g(x) - tf(x)) dx \stackrel{(1)}{=} \\ & = h(c) \int_c^1 (tf(x) - p(x)) dx = \\ & = \int_c^1 h(c) (tf(x) - g(x)) & \stackrel{\substack{h(x) \geq h(c) \\ p(x) \leq tf(x)}}{\leq} \int_c^1 h(x) (tf(x) - p(x)) \Rightarrow \end{aligned}$$

$$\begin{aligned}
&\Rightarrow \int_0^c h(x) p(x) dx - t \int_0^c h(x) f(x) dx \leq \\
&\leq t \int_0^c h(x) f(x) dx - \int_0^c h(x) g(x) dx \Rightarrow \\
&\Rightarrow \int_0^1 h(x) g(x) dx \leq \\
&\leq \frac{\int_0^1 g(x) dx}{\int_0^1 f(x) dx} \int_0^1 h(x) f(x) dx \Rightarrow \int_0^1 h(x) g(x) dx \cdot \int_0^1 f(x) dx \leq \\
&\leq \int_0^1 g(x) dx \cdot \int_0^1 h(x) f(x) dx
\end{aligned}$$

W9. (Solution by the proposer.) Let $S_k = F_2 + F_4 + \dots + F_{2k}$. We claim that $S_k = F_{2k+1} - 1$. Indeed, we have

$$\begin{aligned}
F_1 &= F_2 - F_0 \\
F_3 &= F_4 - F_2 \\
&\vdots \\
F_{2k-3} &= F_{2k-2} - F_{2k-4} \\
F_{2k-1} &= F_{2k} - F_{2k-2}
\end{aligned}$$

Adding up the preceding expressions, yields

$$\sum_{i=1}^k F_{2i-1} = F_{2k} - F_0$$

Then,

$$\sum_{i=1}^k F_{2i} = \sum_{i=1}^{2k} F_i - \sum_{i=1}^k F_{2i-1} = (F_{2k+2} - 1) - F_{2k} = F_{2k+1} - 1$$

on account that

$$\sum_{i=1}^{2k} F_i = F_{2k+2} - 1$$

as can be easily proven by induction. Finally,

$$\begin{aligned} \sum_{k=1}^n \frac{F_{2k}}{(F_{2k+1} - 1)^2} &= \sum_{k=1}^n \frac{F_{2k}}{S_k^2} = 1 + \sum_{k=2}^n \frac{S_k - S_{k-1}}{S_k^2} \\ &\leq 1 + \sum_{k=2}^n \left(\frac{1}{S_{k-1}} - \frac{1}{S_k} \right) = 1 + \frac{1}{S_1} - \frac{1}{S_n} < 1 + \frac{1}{F_2} = 2, \end{aligned}$$

and we are done.

Second solution. Since $F_{2k-1} < F_{2k+1}$, $k \in \mathbb{N}$ then

$$\begin{aligned} \frac{F_{2k}}{(F_{2k+1} - 1)^2} &< \frac{F_{2k}}{(F_{2k-1} - 1)(F_{2k+1} - 1)} = \\ &= \frac{F_{2k+1} - F_{2k-1}}{(F_{2k-1} - 1)(F_{2k+1} - 1)} = \frac{F_{2k+1} - 1 - (F_{2k-1} - 1)}{(F_{2k-1} - 1)(F_{2k+1} - 1)} = \\ &= \frac{1}{F_{2k+1} - 1} - \frac{1}{F_{2k+1} - 1} \end{aligned}$$

and, therefore

$$\begin{aligned} \sum_{k=1}^n \frac{F_{2k}}{(F_{2k+1} - 1)^2} &= \frac{F_2}{(F_3 - 1)^2} + \sum_{k=2}^n \frac{F_{2k}}{(F_{2k+1} - 1)^2} = 1 + \sum_{k=2}^n \frac{F_{2k}}{(F_{2k+1} - 1)^2} < \\ &< 1 + \sum_{k=2}^n \left(\frac{1}{F_{2k+1} - 1} - \frac{1}{F_{2k+1} - 1} \right) = 1 + \left(\frac{1}{F_3 - 1} - \frac{1}{F_{2n+1} - 1} \right) = \\ &= 1 + \left(\frac{1}{2 - 1} - \frac{1}{F_{2n+1} - 1} \right) < 2. \end{aligned}$$

W10. (Solution by the proposer.) Without loss of generality we can assume that $a \geq b \geq c$ from which immediately follows that $a + b \geq a + c \geq b + c$ and

$$\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}.$$

Since the first and the last sequences are sorted in the same way, by applying rearrangement inequality, we get

$$\begin{aligned} 3n \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \right) &\geq 3n \left(\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \right) \\ 3n \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \right) &\geq 3n \left(\frac{b}{a+b} + \frac{c}{b+c} + \frac{a}{c+a} \right) \\ \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} &\geq \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \end{aligned}$$

Adding up the preceding inequalities, yields

$$(6n+1) \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \right) \geq 9n + \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a}$$

from which we obtain

$$\frac{1}{3} \left(\frac{(6n+1)a-b}{b+c} + \frac{(6n+1)b-c}{c+a} + \frac{(6n+1)c-a}{a+b} \right) \geq 3n$$

Taking into account AM-QM inequality, we have

$$\begin{aligned} &\sqrt{\frac{1}{3} \left[\left(\frac{(6n+1)a-b}{b+c} \right)^2 + \left(\frac{(6n+1)b-c}{c+a} \right)^2 + \left(\frac{(6n+1)c-a}{a+b} \right)^2 \right]} \\ &\geq \frac{1}{3} \left(\frac{(6n+1)a-b}{b+c} + \frac{(6n+1)b-c}{c+a} + \frac{(6n+1)c-a}{a+b} \right) \geq 3n \end{aligned}$$

Squaring and arranging terms, yields

$$\left(\frac{(6n+1)a-b}{b+c} \right)^2 + \left(\frac{(6n+1)b-c}{c+a} \right)^2 + \left(\frac{(6n+1)c-a}{a+b} \right)^2 \geq 27n^2$$

from which the statement follows. Equality holds when $a = b = c$ and we are done.

Second solution. Let $m := 3n$. Then

$$\begin{aligned} \sum_{cyc} \left(\frac{(6n+1)a-b}{n(b+c)} \right)^2 \geq 27 &\iff \sum_{cyc} \left(\frac{(6n+1)a-b}{b+c} \right)^2 \geq 27n^2 \iff \\ &\iff \sum_{cyc} \left(\frac{(2m+1)a-b}{b+c} \right)^2 \geq 3m^2 \end{aligned} \quad (1)$$

and we will prove that inequality (1) holds for any $m \in \mathbb{N}$. Since

$$\begin{aligned} \frac{((2m+1)a-b)^2}{b+c} \geq 2m((2m+1)a-b) - m^2(b+c) &\iff \\ &\iff ((2m+1)a-b-m(b+c))^2 \geq 0 \end{aligned}$$

then

$$\sum_{cyc} \left(\frac{(2m+1)a-b}{b+c} \right)^2 \geq \sum_{cyc} \left(\frac{2m((2m+1)a-b)}{b+c} - m^2 \right).$$

Thus, suffices to prove inequality

$$\sum_{cyc} \frac{2m((2m+1)a-b)}{b+c} \geq 6m^2 \iff \sum_{cyc} \frac{(2m+1)a-b}{b+c} \geq 3m \quad (2)$$

Inequality (2) holds because by Cauchy Inequality

$$\begin{aligned} \sum_{cyc} \frac{(2m+1)a-b}{b+c} &= \sum_{cyc} \frac{((2m+1)a-b)^2}{(b+c)((2m+1)a-b)} \geq \\ &\geq \frac{\left(\sum_{cyc} ((2m+1)a-b) \right)^2}{\sum_{cyc} (b+c)((2m+1)a-b)} = \frac{4m^2(a+b+c)^2}{(4m+1)(ab+ac+bc) - (a^2+b^2+c^2)} \end{aligned}$$

and

$$\begin{aligned} \frac{4m^2(a+b+c)^2}{(4m+1)(ab+ac+bc) - (a^2+b^2+c^2)} &\geq 3m \iff \\ &\iff 4m(a+b+c)^2 \geq \end{aligned}$$

$$\geq 3(4m+1)(ab+ac+bc) - 3(a^2+b^2+c^2) \iff$$

$$4m(a+b+c)^2 \geq$$

$$\geq 3(4m+1)(ab+ac+bc) - 3(a+b+c)^2 + 6(ab+ac+bc) \iff$$

$$(4m+3)(a+b+c)^2 \geq 3(4m+3)(ab+ac+bc) \iff (a+b+c)^2 \geq$$

$$\geq 3(ab+ac+bc).$$

Arkady Alt

Third solution. Using the inequality, we have

$$x^2 + y^2 + z^2 \geq \frac{1}{3}(x+y+z)^2$$

$$\begin{aligned} & \left(\frac{(6n+1)a-b}{n(b+c)} \right)^2 + \left(\frac{(6n+1)b-c}{n(c+a)} \right)^2 + \left(\frac{(6n+1)c-a}{n(a+b)} \right)^2 \geq \\ & \geq \frac{1}{3} \left(\frac{(6n+1)a-b}{n(b+c)} + \frac{(6n+1)b-c}{n(c+a)} + \frac{(6n+1)c-a}{n(a+b)} \right)^2 \end{aligned}$$

Using Cauchy-Buniakowski-Schwarz inequality, we get

$$\begin{aligned} & \frac{(6n+1)a-b}{n(b+c)} + \frac{(6n+1)b-c}{n(c+a)} + \frac{(6n+1)c-a}{n(a+b)} \geq \\ & \geq (CBS) \left[\frac{(\sum (6n+1)a - \sum a)^2}{\sum ((6n+1)a-b)(n(b+c))} \right] = \frac{36n^2(\sum a)^2}{n((12n+1)\sum ab - \sum a^2)} \end{aligned}$$

We show that

$$\begin{aligned} & \frac{36n^2(\sum a)^2}{n((12n+1)\sum ab - \sum a^2)} \geq 9 \\ & \frac{36n^2(\sum a)^2}{n((12n+1)\sum ab - \sum a^2)} \geq 9 \Leftrightarrow \end{aligned}$$

$$\Leftrightarrow 4n \left(\sum a \right)^2 \geq (12n+1) \sum ab - \sum a^2 \Rightarrow (4n+1) \left(\sum a^2 - \sum ab \right) \geq 0$$

that is true.

Nicușor Zlota

W11. (Solution by the proposer.) First, we observe that there are numbers satisfying the equality stated, for instance, 1, 2, 4, and 5. To prove the statement we argue by contradiction. Suppose that there exists four positive integers x, y, z, t with $x > y > z > t$ such that

$x^2 + xz - z^2 = y^2 + yt - t^2$ and $p = xy + zt$ is prime. Substituting $x = \frac{p-zt}{y}$ in the expression $x^2 + xz - z^2 = y^2 + yt - t^2$, we get

$$\left(\frac{p-zt}{y} \right)^2 + \left(\frac{p-zt}{y} \right) z - z^2 = y^2 + yt - t^2$$

and reordering terms, yields

$$p(p - 2zt + yz) = (y^2 + z^2)(y^2 + yt - t^2)$$

Since p is prime, then p divides $y^2 + z^2$ or divides $y^2 + yt - t^2$.

- If $p \mid y^2 + z^2$, then $0 < y^2 + z^2 < 2xy < 2(xy + zt) = 2p$ and this implies $y^2 + z^2 = p = xy + zt$ from which follows $y \mid z(z-t)$. Since $xy + zt$ is prime, then $\gcd(y, z) = 1$, and therefore, $p \mid (z-t)$ which is impossible because $0 < z-t < z < y$.
- If $p \mid y^2 + yt - t^2$, then $0 < y^2 + yt - t^2 < 2(xy + zt) = 2p$ and this implies $y^2 + yt - t^2 = p$. That is, $xy + zt = y^2 + yt - t^2 = x^2 - xz - z^2$ from which follows $x \mid z(z+t)$ and $y \mid t(z+t)$. As $\gcd(xy, zt) = 1$, then $x \mid (z+t)$ and $y \mid (z+t)$. Since $0 < z+t < 2x$ and $0 < z+t < 2y$, then $z+t = x$ and $z+t = y$ which is impossible.

The preceding contradictions let us to conclude that $xy + zt$ is composite.

W12. (Solution by the proposer.) The limit equals $4e^\gamma$ where γ denotes the Euler-Mascheroni constant. A calculation shows that

$$O_n = \gamma_{2n} - \frac{1}{2}\gamma_n + \ln 2 + \ln \sqrt{n},$$

where $\gamma_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n$.

Let $y_n = \frac{2O_n}{n}$, $n \geq 1$. We have

$$\begin{aligned} x_n &= \frac{1}{n} \left(1 + \frac{2O_n}{n} \right)^n = \frac{1}{n} e^{\ln(1+x_n)} = \frac{1}{n} e^{n \left(\frac{\ln(1+y_n)-y_n}{y_n^2} \cdot y_n^2 + y_n \right)} = \\ &= \frac{1}{n} e^{\frac{\ln(1+y_n)-y_n}{y_n^2} \cdot n y_n^2 + n y_n} = \frac{1}{n} e^{\frac{\ln(1+y_n)-y_n}{y_n^2} \cdot n y_n^2} \cdot e^{n y_n} \end{aligned}$$

We have $\lim_{n \rightarrow \infty} n y_n^2 = 0$ and, since $\lim_{n \rightarrow \infty} y_n = 0$, we get that

$$\lim_{n \rightarrow \infty} \frac{\ln(1+y_n)-y_n}{y_n^2} = -\frac{1}{2}. \text{ On the other hand,}$$

$$n y_n = 2O_n = 2\gamma_{2n} - \gamma_n + \ln 4 + \ln n$$

and it follows that

$$x_n = e^{\frac{\ln(1+y_n)-y_n}{y_n^2} \cdot n y_n^2} \cdot e^{2\gamma_{2n}-\gamma_n+\ln 4}$$

which implies $\lim_{n \rightarrow \infty} x_n = 4e^\gamma$.

Second solution. Let $h_n := \sum_{k=1}^n \frac{1}{k}$ and $\alpha_n := \frac{2O_n}{n}$. Since

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(1 + \frac{2O_n}{n} \right)^n = \lim_{n \rightarrow \infty} \frac{1}{n} (1 + \alpha_n)^n = \lim_{n \rightarrow \infty} e^{n \ln(1+\alpha_n) - \ln n}$$

we will find

$$\lim_{n \rightarrow \infty} (n \ln(1 + \alpha_n) - \ln n).$$

Noting that $O_n = h_{2n} - \frac{1}{2}h_n$ and $\ln n + \gamma < h_n < \ln(n+1) + \gamma$, where $\gamma \approx 0.5772\dots$ is Euler's constant we obtain

$$2(\ln(2n) + \gamma) - (\ln(n+1) + \gamma) < 2O_n < 2(\ln(2n+1) + \gamma) - (\ln n + \gamma) \iff$$

$$\iff \ln \frac{4n^2}{n+1} + \gamma < 2O_n < \ln \frac{(2n+1)^2}{n} + \gamma \iff$$

$$\iff \frac{1}{n} \left(\ln \frac{4n^2}{n+1} + \gamma \right) < \alpha_n < \frac{1}{n} \left(\ln \frac{(2n+1)^2}{n} + \gamma \right) \quad (1)$$

Since

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(\ln \frac{4n^2}{n+1} + \gamma \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \left(\ln \frac{(2n+1)^2}{n} + \gamma \right) = 0$$

then by Squeeze Principle $\lim_{n \rightarrow \infty} \alpha_n = 0$.

Furthermore, $\lim_{n \rightarrow \infty} n\alpha_n^2 = 0$ as well. Indeed, since

$$\frac{1}{n} \left(\gamma + \ln \frac{4n^2}{n+1} \right)^2 < n\alpha_n^2 < \frac{1}{n} \left(\gamma + \ln \frac{(2n+1)^2}{n} \right)^2$$

and

$$\lim_{n \rightarrow \infty} \frac{\gamma + \ln \frac{4n^2}{n+1}}{\ln 4n} = \lim_{n \rightarrow \infty} \frac{\gamma + \ln \frac{(2n+1)^2}{n}}{\ln 4n} = 1$$

then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(\gamma + \ln \frac{4n^2}{n+1} \right)^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \left(\gamma + \ln \frac{(2n+1)^2}{n} \right)^2 =$$

$$= \lim_{n \rightarrow \infty} \frac{\ln^2 4n}{n} = \lim_{n \rightarrow \infty} \left(\frac{\ln 4n}{\sqrt{n}} \right)^2 = \left(\lim_{n \rightarrow \infty} \frac{\ln n + \ln 4}{\sqrt{n}} \right)^2 = \left(\lim_{n \rightarrow \infty} \frac{\ln n}{\sqrt{n}} \right)^2 = 0.$$

Using inequalities $x - \frac{x^2}{2} < \ln(1+x) < x, x > 0$ and **(1)** we obtain

$$\ln \frac{4n^2}{n(n+1)} + \gamma - \frac{n\alpha_n^2}{2} < n\alpha_n - \frac{n\alpha_n^2}{2} - \ln n < \frac{n \ln(1+\alpha_n) - \ln n}{2} <$$

$$< n\alpha_n - \ln n < \ln \frac{(2n+1)^2}{n^2} + \gamma \implies \ln \frac{4n^2}{n(n+1)} + \gamma - \frac{n\alpha_n^2}{2} <$$

$$< n \ln(1+\alpha_n) - \ln n < \ln \frac{(2n+1)^2}{n^2} + \gamma.$$

Since

$$\lim_{n \rightarrow \infty} \ln \frac{4n^2}{n(n+1)} = \lim_{n \rightarrow \infty} \ln \frac{(2n+1)^2}{n^2} = \ln 4$$

and

$$\lim_{n \rightarrow \infty} n\alpha_n^2 = 0$$

then by Squeeze Principle we obtain

$$\lim_{n \rightarrow \infty} (n \ln(1 + \alpha_n) - \ln n) = \ln 4 + \gamma$$

and, therefore,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(1 + \frac{2O_n}{n} \right)^n = \lim_{n \rightarrow \infty} e^{\ln 4 + \gamma} = 4e^\gamma.$$

Arkady Alt

Third solution. Let $h_n = \sum_{k=1}^n \frac{1}{k}$. Since $O_n = h_{2n} - \frac{1}{2}h_n$ and $\ln n + \gamma < h_n < \ln(n+1) + \gamma$, where $\gamma \approx 0.5772\dots$ is Euler's constant then

$$\ln(2n) + \gamma - \frac{1}{2}(\ln(n+1) + \gamma) < O_n < \ln(2n+1) + \gamma - \frac{1}{2}(\ln n + \gamma) \iff$$

$$\iff \ln\left(\frac{2n}{\sqrt{n+1}}\right) + \frac{\gamma}{2} < O_n < \ln\left(\frac{2n+1}{\sqrt{n}}\right) + \frac{\gamma}{2} \iff$$

$$\iff \ln\left(\frac{2n}{2\sqrt{n}\sqrt{n+1}}\right) < \alpha_n < \ln\left(\frac{2n+1}{2n}\right),$$

where

$$\alpha_n := O_n - \ln(2\sqrt{n}) - \frac{\gamma}{2} = O_n - \frac{1}{2}\ln n - \frac{\gamma}{2} - \ln 2.$$

Hence $O_n = \frac{1}{2}\ln n + \frac{\gamma}{2} + \ln 2 + \alpha_n, n \in \mathbb{N}$ and by Squeeze Principle

$$\lim_{n \rightarrow \infty} \alpha_n = \ln 1 = 0.$$

Note that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(1 + \frac{2O_n}{n} \right)^n = \lim_{n \rightarrow \infty} e^{n \ln \left(1 + \frac{2O_n}{n} \right) - \ln n}.$$

Since $\ln(1+x) = x - \frac{x^2}{2} + o(x^2), x > 0$ then

$$n \ln \left(1 + \frac{2O_n}{n} \right) - \ln n = n \left(\frac{2O_n}{n} + \frac{4O_n^2}{2n^2} + o \left(\frac{O_n^2}{n^2} \right) \right) =$$

$$= 2O_n + \frac{2O_n^2}{n} + o \left(\frac{O_n^2}{n} \right) - \ln n = \ln n + 2 \ln 2 + \gamma + 2\alpha_n + \frac{2O_n^2}{n} + o \left(\frac{O_n^2}{n} \right) - \ln n =$$

$$= 2 \ln 2 + \gamma + 2\alpha_n + \frac{2O_n^2}{n} + o \left(\frac{O_n^2}{n} \right)$$

and, therefore

$$\lim_{n \rightarrow \infty} \left(n \ln \left(1 + \frac{2O_n}{n} \right) - \ln n \right) = 2 \ln 2 + \gamma + \lim_{n \rightarrow \infty} \left(2\alpha_n + \frac{2O_n^2}{n} + o \left(\frac{O_n^2}{n} \right) \right) =$$

$$= 2 \ln 2 + \gamma + \lim_{n \rightarrow \infty} \left(\frac{2O_n^2}{n} + o \left(\frac{O_n^2}{n} \right) \right) = 2 \ln 2 + \gamma$$

because

$$\lim_{n \rightarrow \infty} \frac{O_n^2}{n} = \lim_{n \rightarrow \infty} \frac{\ln^2 n}{n} = 0 \implies \lim_{n \rightarrow \infty} o \left(\frac{O_n^2}{n} \right) = 0.$$

Thus,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(1 + \frac{2O_n}{n} \right)^n = \lim_{n \rightarrow \infty} e^{2 \ln 2 + \gamma} = 4e^\gamma.$$

Arkady Alt

W13. (Solution by the proposer.) The limit equals

$$\begin{cases} e^l, & p = 1 \\ e^{l+2}, & p = 2 \\ \infty, & p \geq 3 \end{cases}$$

We have that

$$\begin{aligned} x_n &= \sum_{k=1}^n \ln \left(a_n + \frac{1}{\sqrt[p]{kn}} \right) = \sum_{k=1}^n \ln \left(1 + \left(a_n - 1 + \frac{1}{\sqrt[p]{kn}} \right) \right) = \\ &= \sum_{k=1}^n \frac{\ln \left(1 + \left(a_n - 1 + \frac{1}{\sqrt[p]{kn}} \right) \right)}{a_n - 1 + \frac{1}{\sqrt[p]{kn}}} \left(a_n - 1 + \frac{1}{\sqrt[p]{kn}} \right) \end{aligned}$$

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W15. (Solution by the proposer.) The inequality is actually symmetric. For instance we can exchange a with b and get

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a} - \left(\frac{b^2}{a+b} + \frac{a^2}{a+c} + \frac{c^2}{c+b} \right) = \sum_{\text{cyc}} \frac{a^2 - b^2}{a+b} = \sum_{\text{cyc}} (a-b) = 0$$

and

$$\sum_{\text{cyc}} \frac{a^3}{b^2 + c^2}$$

is clearly symmetric.

Clearing the denominators in

$$\sum_{\text{cyc}} \frac{2a^2}{a+b} + \frac{(a^2 + b^2 + c^2)^2}{ab^2 + ba^2 + bc^2 + cb^2 + ca^2 + ac^2} \geq \frac{9}{2} \frac{a^2 + b^2 + c^2}{a+b+c}$$

we obtain

$$\sum_{\text{sym}} 2a^7b + 3(ab)^4 \geq \sum_{\text{sym}} (a^6b^2 + a^6bc + 2a^4b^2c^2 + a^3b^3c^2)$$

and the conclusion follows by using the AGM.

Indeed

$$5a^7b + ab^7 \geq 6a^6b^2 \implies \sum_{\text{sym}} a^7b \geq \sum_{\text{sym}} a^6b^2$$

$$36a^7b + b^7c + 6c^7a \geq 43a^6bc \implies \sum_{\text{sym}} a^7b \geq \sum_{\text{sym}} a^6bc$$

$$(ab)^4 + (ac)^4 \geq 2a^4(bc)^2 \implies \sum_{\text{sym}} (ab)^4 \geq \sum_{\text{sym}} a^4(bc)^2$$

$$2(ab)^4 + (bc)^4 + (ca)^4 \geq 4(ab)^3c^2 \implies \sum_{\text{sym}} (ab)^4 \geq \sum_{\text{sym}} (ab)^3c^2$$

and the proof is complete.

Second solution. Noting that

$$\frac{2a^2}{a+b} = \frac{a^2+b^2}{a+b} + \frac{a^2-b^2}{a+b} = \frac{a^2+b^2}{a+b} + a-b$$

and $\sum_{cyc} (a-b) = 0$ we obtain

$$\begin{aligned} & \sum_{cyc} \left(\frac{2a^2}{a+b} + \frac{a^3}{b^2+c^2} \right) \geq \frac{9}{2} \cdot \frac{a^2+b^2+c^2}{a+b+c} \iff \\ & \iff \sum_{cyc} \left(\frac{a^2+b^2}{a+b} + \frac{a^3}{b^2+c^2} \right) \geq \frac{9}{2} \cdot \frac{a^2+b^2+c^2}{a+b+c} \iff \\ & \iff \sum_{cyc} \left(\frac{(a^2+b^2)(a+b+c)}{a+b} + \frac{a^3(a+b+c)}{b^2+c^2} \right) \geq \frac{9}{2} (a^2+b^2+c^2) \iff \\ & \iff \sum_{cyc} \frac{c(a^2+b^2)}{a+b} + \sum_{cyc} (a^2+b^2) + \sum_{cyc} \frac{a^4}{b^2+c^2} + \sum_{cyc} \frac{a^3(b+c)}{b^2+c^2} \geq \\ & \geq \frac{9}{2} (a^2+b^2+c^2) \iff \\ & \iff \sum_{cyc} \frac{c(a^2+b^2)}{a+b} + 2(a^2+b^2+c^2) + \sum_{cyc} \frac{a^4}{b^2+c^2} + \sum_{cyc} \frac{a^3(b+c)}{b^2+c^2} \geq \\ & \geq \frac{9}{2} (a^2+b^2+c^2) \iff \\ & \iff \sum_{cyc} \frac{c(a^2+b^2)}{a+b} + \sum_{cyc} \frac{a^4}{b^2+c^2} + \sum_{cyc} \frac{a^3(b+c)}{b^2+c^2} \geq \frac{5}{2} (a^2+b^2+c^2). \end{aligned}$$

Since

$$\sum_{cyc} \frac{c(a^2+b^2)}{a+b} + \sum_{cyc} \frac{a^3(b+c)}{b^2+c^2} = \sum_{cyc} \frac{a(b^2+c^2)}{b+c} + \sum_{cyc} \frac{a^3(b+c)}{b^2+c^2} =$$

$$= \sum_{cyc} a \left(\frac{b^2 + c^2}{b + c} + \frac{a^2(b + c)}{b^2 + c^2} \right) \geq \sum_{cyc} a \cdot 2 \sqrt{\frac{b^2 + c^2}{b + c} \cdot \frac{a^2(b + c)}{b^2 + c^2}} = 2 \sum_{cyc} a^2$$

and by Cauchy Inequality

$$\sum_{cyc} \frac{a^4}{b^2 + c^2} \geq \frac{(a^2 + b^2 + c^2)^2}{\sum_{cyc} (b^2 + c^2)} = \frac{a^2 + b^2 + c^2}{2}$$

then

$$\begin{aligned} \sum_{cyc} \frac{c(a^2 + b^2)}{a + b} + \sum_{cyc} \frac{a^4}{b^2 + c^2} + \sum_{cyc} \frac{a^3(b + c)}{b^2 + c^2} &\geq 2(a^2 + b^2 + c^2) + \\ &+ \frac{a^2 + b^2 + c^2}{2} = \frac{5}{2}(a^2 + b^2 + c^2). \end{aligned}$$

Arkady Alt

W16. (Solution by the proposer.) The inequality is implied by

$$\sin t (\cos^3 t + \sin t) \geq \left(\frac{2t}{\pi} \right)^{\cos^2 t}, \quad 0 \leq t \leq \frac{\pi}{4}$$

By the weighted-AGM we have

$$\sin t \cos t \cdot \cos^2 t + 1 \cdot \sin^2 t \geq \left(\frac{1}{2} \sin(2t) \cdot 1 \right)^{\cos^2 t}$$

Moreover by $\sin(2t) \geq (4t)/\pi$ for $0 \leq 2t \leq \frac{\pi}{2}$ we get

$$\sin t \cos t \cdot \cos^2 t + 1 \cdot \sin^2 t \geq \left(\frac{1}{2} \sin(2t) \cdot 1 \right)^{\cos^2 t} \geq \left(\frac{2t}{\pi} \right)^{\cos^2 t}$$

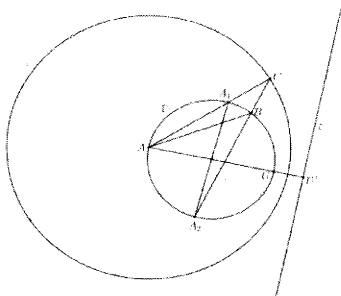
whence the inequality.

W17. (Solution by the proposer.) The case $p = 1$ is trivial because the solution of the recurrence $a_{n+1} = \frac{a_n}{1+a_n}$ is exactly $a_n = \frac{1}{n}$ setting to zero the terms of the series.

If $p \neq 1$, the proof amounts to find the first two terms of the expansion of a_n .

We observe that $a_{n+1} < a_n$ and $a_n > 0$. It follows the existence of the limit

As a result, the maximal value of $\frac{MA}{MC}$ is attained at $J(A_2)$. But $J(A_2) = B$, so we are done.



W21. (Solution by the proposer.) The identity

$$2(b^2 + c^2) = 4m_a^2 + a^2 \quad (1)$$

holds. Taking $AM - GM$ inequality, we have

$$4m_a^2 + a^2 = \frac{4}{3}m_a^2 + \frac{4}{3}m_a^2 + \frac{4}{3}m_a^2 + a^2 \geq 4\sqrt[4]{\frac{64}{27}m_a^6a^2}$$

with equality if and only if $\frac{4}{3}m_a^2 = a^2$, from where

$$4m_a^2 + a^2 \geq \frac{8\sqrt[4]{12}m_a\sqrt{am_a}}{3} \quad (2)$$

From (1) and (2) we obtain the inequality

$$b^2 + c^2 \geq \frac{4\sqrt[4]{12}m_a\sqrt{am_a}}{3} \quad (3)$$

with equality if and only if $b^2 + c^2 = 2a^2$. Taking (3) into account the inequality from hypothesis's follows. The equality holds if and only if ABC is equilateral triangle.

Second solution. By replacing (a, b, c, m_a, m_b, m_c) in original inequality with $\left(\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}, \frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$ (Median Transformation) we obtain equivalent inequality

$$\begin{aligned}
& \sum_{cyc} \frac{\left(\left(\frac{2m_b}{3} \right)^2 + \left(\frac{2m_c}{3} \right)^2 \right)^2}{\left(\frac{a}{2} \right)^3} \geq \frac{32\sqrt{3}}{9} \left(\frac{2m_a}{3} + \frac{2m_b}{3} + \frac{2m_c}{3} \right) \Leftrightarrow \\
& \Leftrightarrow \sum_{cyc} \frac{\frac{16}{81} (m_b^2 + m_c^2)^2}{\frac{a^3}{8}} \geq \frac{32\sqrt{3}}{9} \cdot \frac{2}{3} (m_a + m_b + m_c) \Leftrightarrow \\
& \Leftrightarrow \sum_{cyc} \frac{(m_b^2 + m_c^2)^2}{a^3} \geq \frac{3\sqrt{3}}{2} (m_a + m_b + m_c) \Leftrightarrow \\
& \Leftrightarrow \sum_{cyc} \frac{(4a^2 + b^2 + c^2)^2}{a^3} \geq 24\sqrt{3} (m_a + m_b + m_c) \quad (1)
\end{aligned}$$

$$(\text{since } m_b^2 + m_c^2 = \frac{4a^2 + b^2 + c^2}{4})$$

Applying inequality $\frac{u^2}{v} \geq 2u - v$, $v > 0$ for $u = 4a^2 + b^2 + c^2$ and $v = 6a^2$ we obtain

$$\begin{aligned}
& \frac{(4a^2 + b^2 + c^2)^2}{a^3} = \frac{6}{a} \cdot \frac{(4a^2 + b^2 + c^2)^2}{6a^2} \geq \\
& \geq \frac{6(2(4a^2 + b^2 + c^2) - 6a^2)}{a} = \frac{12(a^2 + b^2 + c^2)}{a}.
\end{aligned}$$

Therefore,

$$\sum_{cyc} \frac{(4a^2 + b^2 + c^2)^2}{a^3} \geq 12(a^2 + b^2 + c^2) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right).$$

Since

$$\begin{aligned}
m_a + m_b + m_c & \leq \sqrt{3(m_a^2 + m_b^2 + m_c^2)} = \sqrt{3 \cdot \frac{3}{4}(a^2 + b^2 + c^2)} = \\
& = \frac{3}{2} \sqrt{a^2 + b^2 + c^2}
\end{aligned}$$

then

$$24\sqrt{3}(m_a + m_b + m_c) \leq 24\sqrt{3} \cdot \frac{3}{2}\sqrt{a^2 + b^2 + c^2} = 36\sqrt{3}\sqrt{a^2 + b^2 + c^2}.$$

Thus, suffices to prove inequality

$$\begin{aligned} 12(a^2 + b^2 + c^2) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) &\geq 36\sqrt{3}\sqrt{a^2 + b^2 + c^2} \iff \\ \iff \sqrt{a^2 + b^2 + c^2} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) &\geq 3\sqrt{3} \iff \\ \iff (a^2 + b^2 + c^2) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)^2 &\geq 27. \end{aligned}$$

Since by AM-GM Inequality $a^2 + b^2 + c^2 \geq 3\sqrt[3]{a^2b^2c^2}$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{3}{\sqrt[3]{abc}}$

then $(a^2 + b^2 + c^2) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)^2 \geq 27$ and, therefore, we have

$$\begin{aligned} \sum_{cyc} \frac{(4a^2 + b^2 + c^2)^2}{a^3} &\geq 12(a^2 + b^2 + c^2) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq \\ &\geq 36\sqrt{3}\sqrt{a^2 + b^2 + c^2} \geq 24\sqrt{3}(m_a + m_b + m_c). \end{aligned}$$

Arkady Alt

W22. (Solution by the proposer.) If we note $\frac{AO}{OC} = k$, then $\frac{AO}{AC} = \frac{k}{k+1}$ and $\frac{CO}{AC} = \frac{1}{k+1} \cdot BCD$. From $\frac{AO}{AC} = \frac{OM}{CB} = \frac{OQ}{CD}$ we have

$$OM = \frac{k}{k+1}BC \quad (1)$$

$$OQ = \frac{k}{k+1}CD \quad (2)$$

and from $\frac{CO}{CA} = \frac{ON}{AB} = \frac{OP}{AD}$ we have

$$ON = \frac{1}{k+1}AB \quad (3)$$

and

$$OP = \frac{1}{k+1}AD \quad (4)$$

$$x = y (\ln y + \ln \ln y + o(\ln \ln y))$$

we deduce

$$p_n = n (\ln n + \ln \ln n + o(\ln \ln n)) \quad (*)$$

for $n \rightarrow +\infty$

$$\left| \frac{e^{i \ln(p_n)}}{p_n} - (n \ln n)^{i-1} \right| \leq \sqrt{2} \frac{1}{n^2 \ln^2 n} |p_n - n \ln n|$$

with (*) we get

$$\frac{e^{i \ln(p_n)}}{p_n} - (n \ln n)^{i-1} = O_{n \rightarrow +\infty} \left(\frac{\ln \ln n}{n \ln^2 n} \right)$$

since $\sum_n \frac{\ln \ln n}{n \ln^2 n}$ converge, $\sum_{n>0} \frac{e^{i \ln(p_n)}}{p_n}$ and $\sum_{n>0} (n \ln n)^{i-1}$ are same nature.

With integral and series we have

$$\int_2^{n+1} (t \ln t)^{i-1} dt = \int_2^n (t \ln t)^{i-1} dt + (n \ln n)^{i-1} + v_n$$

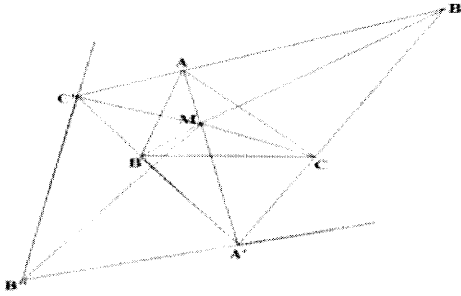
with $|v_n| \leq \frac{K}{n^2 \ln n}$, then

$$\sum_{n=2}^N (n \ln n)^{i-1} = \int_2^{n+1} (t \ln t)^{i-1} dt = \sum_{n=2}^N v_n$$

has a finite limit when $n \rightarrow \infty$. This prove $\sum_{n \geq 1} \frac{e^{i \ln(p_n)}}{p_n}$ converge

W26. (Solution by the proposer.) We constructing the perpendiculars on MA, MB, MC in the points A, B, C . These perpendiculars meet in the points A', B', C' , see the figure. In the same way, in the points A', B', C' we construct the perpendiculars on MA', MB', MC' which meet in A'', B'', C'' .

We calculate the area of the triangle $MA'C'$ in two ways thus:



$$\frac{MA' \cdot MC' \cdot \sin A'MC'}{2} = \frac{MB \cdot A'C'}{2} = \frac{MB \cdot MB'' \cdot \sin A'MC'}{2}$$

So, we deduce

$$MB'' = \frac{MA' \cdot MC'}{MB}$$

Similarly, we obtain:

$$MA'' = \frac{MB' \cdot MC'}{MA} \cdot MC'' = \frac{MA' \cdot MB'}{MC}$$

From Erdős-Mordell's inequality applied in the triangle $A''B''C''$ for the point M, we have:

$$MA'' + MB'' + MC'' \geq 2(MA' + MB' + MC')$$

so

$$\frac{MB' \cdot MC'}{MA} + \frac{MA' \cdot MC'}{MB} + \frac{MB' \cdot MA'}{MC} \geq 2(MA' + MB' + MC')$$

Since $MA' = 2R_a$, $MB' = 2R_b$, $MC' = 2R_c$, it follows:

$$\frac{R_b R_c}{MA} + \frac{R_a R_c}{MB} + \frac{R_a R_b}{MC} \geq R_a + R_b + R_c$$

Therefore, the inequality of the statement.

Second solution. We will use for the radii of circumcircle of MBC , MCA , MAB another notation

ρ_a, ρ_b, ρ_c and standard notation R_a, R_b, R_c for distances MA, MB, MC , respectively.

Also denote via d_a, d_b, d_c distances from M to BC, CA, AB .

Thus, inequality to prove is

$$\frac{1}{\rho_a} + \frac{1}{\rho_b} + \frac{1}{\rho_c} \leq \frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_c}. \quad (1)$$

Since $[BMC] = \frac{ad_a}{2}$ and

$$4\rho_a [BMC] = aR_bR_c \iff 2\rho_a \cdot ad_a = aR_bR_c \iff \frac{1}{\rho_a} = \frac{2d_a}{R_bR_c}$$

$$\text{then } (1) \iff \sum_{cyc} \frac{2d_a}{R_bR_c} \leq \sum_{cyc} \frac{1}{R_a}.$$

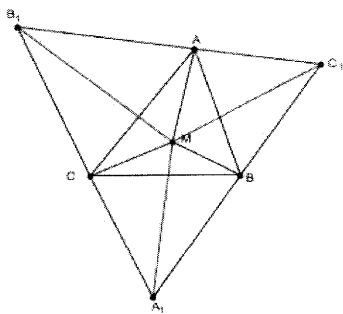


Figure 1

Let us draw throughout vertexes A, B, C respectively three lines perpendicularly to MA, MB, MC respectively. Three points of intersection of these lines determine triangle $A_1B_1C_1$

($A_1B_1 \perp MC, B_1C_1 \perp MA, C_1A_1 \perp MB$) with

$d_{a_1} := R_a = MA, d_{b_1} := R_b = MB, d_{c_1} := R_c = MC$ as distances from M to B_1C_1, C_1A_1, A_1B_1 respectively and with distances between M and vertexes of $\triangle A_1B_1C_1$: $R_{a_1} := MA_1, R_{b_1} := MB_1, R_{c_1} := MC_1$. (Pic.1).

Since $R_{a_1}, R_{b_1}, R_{c_1}$ are diameters of the circumcircles for quadrilaterals MCA_1B, MAB_1C, MBC_1A , respectively, then, by Sine-Theorem, we have

$$R_{a_1} = \frac{a}{\sin \angle BMC} = \frac{aR_bR_c}{R_bR_c \sin \angle BMC} = \frac{aR_bR_c}{ad_a} = \frac{R_bR_c}{d_a}$$

and, similarly,

$$R_{b_1} = \frac{R_cR_a}{d_b}, R_{c_1} = \frac{R_aR_b}{d_c}.$$

Since

$$\sum_{cyc} \frac{2d_a}{R_b R_c} = 2 \sum_{cyc} \frac{1}{R_{a_1}}$$

and

$$\sum_{cyc} \frac{1}{R_a} = \sum_{cyc} \frac{1}{d_{a_1}}$$

then (1) $\iff 2 \sum_{cyc} \frac{1}{R_{a_1}} \leq \sum_{cyc} \frac{1}{d_{a_1}}$.

Thus, suffices to prove that in any triangle ABC with interior point M and R_a, R_b, R_c as distances from point M to vertices A, B, C , respectively, and d_a, d_b, d_c distances from M to BC, CA, AB holds inequality

$$2 \sum_{cyc} \frac{1}{R_a} \leq \sum_{cyc} \frac{1}{d_a}. \quad (2)$$

Lemma. Let P_a and P_b be involutions of P with respect to a and b respectively (that is $PP_a \perp a$, $PP_b \perp b$, $P_a M \cdot PM = P_b N \cdot PN = 1$)

Then $P_a P_b \perp PA$ and $PE = \frac{1}{PA}$ where E is intersection point of $P_a P_b$ and PA .

Proof.

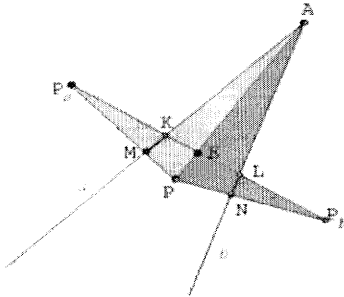


Figure 2

Let $P_a E_1$ and $P_b E_2$ be perpendiculars from P_a and P_b to $\overleftrightarrow{PA}$ respectively ($E_1, E_2 \in \overleftrightarrow{PA}$). Since $\angle PP_a E_1 = \angle PAM$ and $\angle PP_b E_2 = \angle PAN$ (as the angles which constructed by mutually perpendicular sides) then we have $\triangle PP_a E_1 \sim \triangle PAM$ and $\triangle PP_b E_2 \sim \triangle PAN$ and from these similarity follows

$$\frac{PE_1}{PP_a} = \frac{PM}{PA} \iff \frac{PE_1}{\frac{1}{d_a}} = \frac{d_a}{PA} \iff PE_1 = \frac{1}{PA}$$

and

$$\frac{PE_2}{PP_b} = \frac{PN}{PA} \iff \frac{PE_2}{\frac{1}{d_b}} = \frac{d_b}{PA} \iff PE_2 = \frac{1}{PA}.$$

Hence, $PE_1 = PE_2$ and $E := E_1 = E_2$ is intersection point of P_aP_b with PA and $PE = \frac{1}{R_A}$.

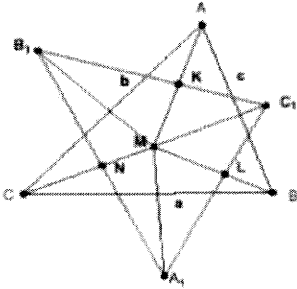


Figure 3

Let A_1, B_1, C_1 be involution points for M with respect to lines $\overleftrightarrow{BC}, \overleftrightarrow{CA}, \overleftrightarrow{AB}$ respectively. Let $R'_a = MA_1 = \frac{1}{d_a}, R'_b = MB_1 = \frac{1}{d_b}, R'_c = MC_1 = \frac{1}{d_c}$ and d'_a, d'_b, d'_c be distances from M to sides B_1C_1, C_1A_1, A_1B_1 . Since by lemma $d'_a = \frac{1}{R_a}, d'_b = \frac{1}{R_b}, d'_c = \frac{1}{R_c}$ then replacing $(R_a, R_b, R_c, d_a, d_b, d_c)$ in Erdős-Mordell Inequality $R_a + R_b + R_c \geq 2(d_a + d_b + d_c)$ with

$$(R'_a, R'_b, R'_c, d'_a, d'_b, d'_c)$$

we obtain

$$\sum_{cyc} R'_a \geq 2 \sum_{cyc} d'_a \iff (2)$$

W27. (Solution by the proposer.) By the AM-GM inequality

$$a_1 + \frac{1}{\prod_{j=1}^k a_j} \geq 2 \sqrt{\frac{1}{\prod_{j=1}^k a_j}} = \frac{2}{\sqrt{\prod_{j=2}^k a_j}}$$

therefore the left-hand side of the proposed inequality is

$$LHS \geq \sqrt[n]{2} \sum_{cyclic} \frac{1}{\sqrt[2n]{\prod_{j=1}^{k-1} a_j}}.$$

Now, since function $f(x) = \frac{1}{\sqrt[2n]{x}}$ is convex for $x > 0$ the conclusion follows by Jensen inequality.

Second solution. Let $A := \prod_{j=1}^k a_j$ and $b_i := \frac{A}{a_i}, i = 1, 2, \dots, k$. Since

$$\sum_{cyc} \prod_{j=1}^{k-1} a_j = k \iff \sum_{i=1}^k b_i = k$$

and

$$\prod_{i=1}^k b_i = \prod_{i=1}^k \frac{A}{a_i} = \frac{A^k}{\prod_{j=1}^k a_j} = A^{k-1}$$

by AM-GM Inequality we obtain

$$k = \sum_{i=1}^k b_i \geq k \sqrt[k]{\prod_{i=1}^k b_i} = k \sqrt[k]{A^{k-1}} \iff A \leq 1.$$

From the other hand

$$\begin{aligned} \sum_{cyc} \sqrt[n]{a_1 + \frac{1}{\prod_{j=1}^k a_j}} &= \sum_{i=1}^k \sqrt[n]{a_i + \frac{1}{A}} \geq \\ &\geq \sum_{i=1}^k \sqrt[n]{a_i + \frac{1}{A}} \geq \sum_{i=1}^k \sqrt[n]{2 \sqrt{\frac{a_i}{A}}} = \sum_{i=1}^k \sqrt[n]{\frac{2}{\sqrt{b_i}}} = \sqrt[n]{2} \sum_{i=1}^k \frac{1}{\sqrt[2n]{b_i}}. \end{aligned}$$

And again applying AM-GM Inequality we obtain

$$\sum_{i=1}^k \frac{1}{\sqrt[2n]{b_i}} \geq k \sqrt[k]{\prod_{j=1}^k \frac{1}{\sqrt[2n]{b_j}}} = k \sqrt[2nk]{\frac{1}{\prod_{j=1}^k b_j}} = k \sqrt[2nk]{\frac{1}{A^{k-1}}} \geq k$$

because $A \leq 1$. Thus,

$$\sum_{cyc} \sqrt[n]{a_1 + \frac{1}{\prod_{j=1}^k a_j}} \geq k \sqrt[n]{2}.$$

Arkady Alt

W28. (Solution by the proposer.) The roots of characteristic polynomial equation $x^2 - x + \frac{1}{2} = 0$ are $\alpha = \frac{1+i}{2}$ and $\beta = \frac{1-i}{2}$, and taking into account the initial values of the sequence, it follows that $x_n = \alpha^n + \beta^n$, so the proposed series reads may be written as

$$\sum_{n=1}^{\infty} \frac{\alpha^n + \beta^n}{n+2}.$$

The generating function for $(x_n)_{n \geq 0}$ is $F(x) = \sum_{n=0}^{\infty} x_n x^n = \frac{2-x}{\frac{x^2}{2} - x + 1}$.

Function $F(z)$ is analytic for $z \in C$ with $|z| < \sqrt{2}$.

To find the proposed series $\sum_{n=1}^{\infty} \frac{x_n}{n+2} = \sum_{n=1}^{\infty} \frac{\alpha^n + \beta^n}{n+2}$, we consider the function $x \cdot F(x) = \frac{2x-x^2}{\frac{x^2}{2}-x+1}$, and its integral

$$\int \frac{2x-x^2}{\frac{x^2}{2}-x+1} dx = -2(x + 2 \tan^{-1}(1-x)) = G(x)$$

$$\text{so } \sum_{n=1}^{\infty} \frac{x_n x^{n+2}}{n+2} = G(x) - G(0) = \pi - 2(x + 2 \tan^{-1}(1-x)).$$

$$\text{Therefore, } \sum_{n=1}^{\infty} \frac{x_n}{n+2} = G(1) - G(0) = \pi - 2(1 + 2 \tan^{-1}(1)) = \pi - 2.$$

Second solution. Note that

$$x_{n+2} = x_{n+1} - \frac{1}{2}x_n \iff$$

$$\iff (\sqrt{2})^{n+2} x_{n+2} - 2 \cdot \frac{1}{\sqrt{2}} (\sqrt{2})^{n+1} x_{n+1} + (\sqrt{2})^n x_n = 0 \iff$$

$$\iff (\sqrt{2})^{n+2} x_{n+2} - 2 \cdot \cos \frac{\pi}{4} (\sqrt{2})^{n+1} x_{n+1} + (\sqrt{2})^n x_n = 0 \iff$$

$$\iff (\sqrt{2})^n x_n = c_1 \cos \frac{n\pi}{4} + c_2 \sin \frac{n\pi}{4}, n \in \mathbb{N} \cup \{0\}.$$

Since $x_0 = 2 \implies c_1 = 2$ and

$$x_1 = 1 \iff (\sqrt{2}) = 2 \cos \frac{\pi}{4} + c_2 \sin \frac{\pi}{4} \iff c_2 = 0 \text{ then}$$

$$x_n = \frac{2 \cos \frac{n\pi}{4}}{(\sqrt{2})^n}, n \in \mathbb{N} \cup \{0\} \text{ and}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{x_n}{n+2} &= \sum_{n=0}^{\infty} \frac{x_n}{n+2} - 1 = 2 \sum_{n=0}^{\infty} \frac{\cos \frac{n\pi}{4}}{(\sqrt{2})^n (n+2)} - 1 = \\ &= 4 \sum_{n=0}^{\infty} \frac{\cos \frac{n\pi}{4}}{(\sqrt{2})^{n+2} (n+2)} - 1 \end{aligned}$$

Let $S(x) := \sum_{n=0}^{\infty} \frac{\cos \frac{n\pi}{4} x^{n+2}}{n+2}$, where $x \in (0, 1)$. Then

$$\sum_{n=1}^{\infty} \frac{x_n}{n+2} = 4S\left(\frac{1}{\sqrt{2}}\right) - 1$$

and $S'(x) = x \sum_{n=0}^{\infty} x^n \cos \frac{n\pi}{4}$, where series $\sum_{n=1}^{\infty} \cos \frac{n\pi}{4} x^n$ converges absolutely

$$\sum_{n=1}^{\infty} \left| \cos \frac{n\pi}{4} x^n \right| \leq \sum_{n=1}^{\infty} |x|^n = \frac{|x|}{1-|x|}$$

Noting that

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\cos \frac{n\pi}{4} + i \sin \frac{n\pi}{4} \right) x^n &= \sum_{n=0}^{\infty} e^{\frac{in\pi}{4}} x^n = \sum_{n=0}^{\infty} \left(e^{\frac{i\pi}{4}} x \right)^n = \\ &= \frac{1}{1 - e^{\frac{i\pi}{4}} x} \end{aligned}$$

(since $\left| e^{\frac{i\pi}{4}} x \right| < 1$) and

$$\begin{aligned}\frac{1}{1 - e^{\frac{i\pi}{4}} x} &= \frac{1}{1 - \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) x} = \frac{1}{1 - \frac{x}{\sqrt{2}} - i \frac{x}{\sqrt{2}}} = \\ &= \frac{\sqrt{2}}{\sqrt{2} - x - ix} = \frac{\sqrt{2}(\sqrt{2} - x + ix)}{(\sqrt{2} - x)^2 + x^2}\end{aligned}$$

we obtain

$$S'(x) = x \sum_{n=0}^{\infty} \cos \frac{n\pi}{4} x^n = \frac{\sqrt{2}(\sqrt{2} - x)x}{(\sqrt{2} - x)^2 + x^2} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}(x^2 - \sqrt{2}x + 1)}.$$

Since $\sum_{n=1}^{\infty} \frac{x_n}{n+2} = 4S\left(\frac{1}{\sqrt{2}}\right) - 1$ and

$$S\left(\frac{1}{\sqrt{2}}\right) = \int_0^{1/\sqrt{2}} \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}(x^2 - \sqrt{2}x + 1)}\right) dx = \frac{\pi}{4} - \frac{1}{2}$$

then

$$\sum_{n=1}^{\infty} \frac{x_n}{n+2} = 4\left(\frac{\pi}{4} - \frac{1}{2}\right) - 1 = \pi - 3.$$

Arkady Alt

W29. (Solution by the proposer.) Since $x + y + z = 1$, the given inequality may be written as

$$\sum_{cyclic} \sqrt{\frac{x^3 + 1}{x^2 - x + 1}} \leq 3\sqrt{2}.$$

Now, using that $x^3 + 1 = (x + 1)(x^2 - x + 1)$, the inequality becomes

$\sum_{cyclic} \sqrt{x + 1} \leq 3\sqrt{2}$. Finally, since function $f(x) = \sqrt{x + 1}$ is concave because

$f''(x) = \frac{-1}{4(x + 1)^{3/2}} < 0$, by Jensen's inequality, we have

$$\frac{\sum_{cyclic} \sqrt{x + 1}}{3} \leq \sqrt{\frac{x + y + z}{3} + 1} = \sqrt{2}$$

from where the result follows.

Second solution. We have

$$\sum_{cyc} \sqrt{\frac{x^3+1}{x^2+y+z}} = \sum_{cyc} \sqrt{\frac{x^3+1}{x^2-x+1}} = \sum_{cyc} \sqrt{x+1}$$

and since

$$a+b+c \leq \sqrt{3(a^2+b^2+c^2)}, a, b, c \in \mathbb{R}$$

then for $a = \sqrt{x+1}, b = \sqrt{y+1}, c = \sqrt{z+1}$ we obtain

$$\sum_{cyc} \sqrt{x+1} \leq \sqrt{3 \sum_{cyc} (\sqrt{x+1})^2} = \sqrt{3(x+y+z+3)} = \sqrt{3 \cdot 4} = 2\sqrt{3}.$$

Arkady Alt

Third solution. We have

$$\begin{aligned} \sum \sqrt{\frac{x^3+1}{x^2+y+z}} &= \sum \sqrt{\frac{x^3+1}{x^2+1-x}} = \sum \sqrt{x+1} \leq \\ &\leq \sqrt{3(x+y+z+3)} = 2\sqrt{3} < 3\sqrt{2} \end{aligned}$$

Nicușor Zlota

W30. (Solution by the proposer.) We have $x_2 = 3$ so

$$x_0 x_2 = x_1^2 + x_1 + 1$$

and we prove by induction the relation

$$x_n x_{n+2} = x_{n+1}^2 + x_{n+1} + 1$$

We suppose that

$$x_{n-1} x_{n+1} = x_n^2 + x_n + 1$$

so

$$x_n x_{n+2} = x_n (5x_{n+1} - x_n - 1) = 5x_n x_{n+1} - x_n^2 - x_n =$$

$$= 5x_n x_{n+1} + 1 - (x_n^2 + x_n + 1) = 5x_n x_{n+1} + 1 - x_{n-1} x_{n+1} =$$

$$= x_{n+1} (5x_n - x_{n-1}) + 1 = x_{n+1} (x_{n+1} + 1) + 1 = x_{n+1}^2 + x_{n+1} + 1$$

Using the Hölder's inequality we get

$$x_n x_m x_p x_{n+2} x_{m+2} x_{p+2} =$$

$$= (x_{n+1}^2 + x_{n+1} + 1) (x_{m+1} + 1 + x_{m+1}^2) (1 + x_{p+1}^2 + x_{p+1}) \geq$$

$$\geq \left(\sqrt[3]{x_{n+1}^2 x_{m+1}} + \sqrt[3]{x_{p+1}^2 x_{n+1}} + \sqrt[3]{x_{m+1}^2 x_{p+1}} \right)^3$$

W31. (Solution by the proposer.) Using the Jensen's inequality we have

$$\begin{aligned} \left(\sum \log_{a_1} a_1 a_2 \right)^\lambda &\geq n \left(\frac{1}{n} \sum \log_{a_1} a_1 a_2 \right)^\lambda = n \left(\frac{1}{n} \sum (\log_{a_1} a_1 + \log_{a_1} a_2) \right)^\lambda = \\ &= n \left(\frac{1}{n} \left(n + \sum \log_{a_1} a_2 \right) \right)^\lambda = n \left(1 + \sqrt[n]{\prod \log_{a_1} a_2} \right)^\lambda = n 2^\lambda \end{aligned}$$

Second solution. Let $x_k := \log_{a_k} a_{k+1}, k = 1, 2, \dots, n-1$ and $x_n := \log_{a_{n+1}} a_1$. Then $x_k > 0, k = 1, 2, \dots, n$,

$$\prod_{k=1}^n x_k = \prod_{cyc} \log_{a_1} a_2 = 1$$

and original inequality becomes

$$\sum_{k=1}^n (1 + x_k)^\lambda \geq n 2^\lambda. \quad (1)$$

I. First we will prove that for any $\lambda \in [0, \infty)$ inequality (1) holds.

Case $\lambda = 0$ is trivial. For any $\lambda \in (0, \infty)$ we have:

1. By AM-GM Inequality

$$\sum_{k=1}^n (1 + x_k)^\lambda \geq n \left(\prod_{k=1}^n (1 + x_k)^\lambda \right)^{\frac{1}{n}} = n \left(\prod_{k=1}^n (1 + x_k) \right)^{\frac{\lambda}{n}};$$

2. Also $1 + x_k \geq 2\sqrt{x_k}$, $k = 1, 2, \dots, n$.

Hence,

$$\begin{aligned} \sum_{k=1}^n (1 + x_k)^\lambda &\geq n \left(\prod_{k=1}^n (1 + x_k) \right)^{\frac{\lambda}{n}} \geq n \left(\prod_{k=1}^n 2\sqrt{x_k} \right)^{\frac{\lambda}{n}} = \\ &= n \left(2^n \sqrt[n]{\prod_{k=1}^n x_k} \right)^{\frac{\lambda}{n}} = n (2^n)^{\frac{\lambda}{n}} = n 2^\lambda. \end{aligned}$$

II. Now consider case $\lambda < 0$.

We will show that for any such λ there are $n \in \mathbb{N}$ and $x_1, x_2, \dots, x_n > 0$ such that

$x_1 x_2 \dots x_n = 1$ and

$$\sum_{k=1}^n (1 + x_k)^\lambda < n 2^\lambda \iff \sum_{k=1}^n \frac{1}{(1 + x_k)^p} < \frac{n}{2^p},$$

where $p := -\lambda > 0$.

For any $p > 0$ let $n > 2^p$ and let $x_k = a > 0$, $k = 1, 2, \dots, n-1$, $x_n = \frac{1}{a^{n-1}}$.

Then $1 < \frac{n}{2^p}$ and

$$\sum_{k=1}^n \frac{1}{(1 + x_k)^p} = \frac{n-1}{(1+a)^p} + \frac{a^{(n-1)p}}{(1+a^{n-1})^p}.$$

Since

$$\lim_{a \rightarrow \infty} \left(\frac{n-1}{(1+a)^p} + \frac{a^{(n-1)p}}{(1+a^{n-1})^p} \right) = 1$$

then

$$\frac{n-1}{(1+a)^p} + \frac{a^{(n-1)p}}{(1+a^{n-1})^p} < \frac{n}{2^p}$$

for big enough a .

Simple particular case:

Let $n = 3$ and $\lambda = -1$ then for $x_1 = x_2 = 2$, $x_3 = 1/4$ we obtain

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \frac{1}{1+x_3} = \frac{2}{1+2} + \frac{1}{1+1/4} = \frac{22}{15} < \frac{3}{2} = 3 \cdot 2^\gamma.$$

Arkady Alt

W32. (Solution by the proposer.) The function

$$f(x) = x \ln(1-x)$$

is concave because

$$f''(x) = -\frac{1}{1-x} - \frac{1}{(1-x)^2} < 0$$

Using the Popoviciu's inequality

$$f(x) + f(y) + f(z) + 3f\left(\frac{x+y+z}{3}\right) \leq 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{y+z}{2}\right) + 2f\left(\frac{z+x}{2}\right)$$

we obtain the desired result.

Second solution. Note that

$$\begin{aligned} 8 \prod_{cyc} (1-x)^x &\leq 3 \prod_{cyc} (2-y-z)^{y+z} \iff 8 \prod_{cyc} (1-x)^x \leq \\ &\leq 3 \prod_{cyc} (1+x)^{1-x} \iff \\ &\iff \prod_{cyc} (1-x^2)^x \leq \frac{3}{8} \prod_{cyc} (1+x). \end{aligned}$$

Since by AM-GM Inequality

$$\prod_{cyc} (1-x^2)^x \leq \sum_{cyc} x(1-x^2) = 1 - (x^3 + y^3 + z^3)$$

suffices to prove inequality

$$1 - (x^3 + y^3 + z^3) \leq \frac{3}{8} (1+x)(1+y)(1+z). \quad (1)$$

Let $p := xy + yz + zx$, $q := xyz$. Then

$$x^3 + y^3 + z^3 = 1 + 3q - 3p,$$

$$(1+x)(1+y)(1+z) = 2+p+q$$

and inequality (1) becomes

$$1 - (1+3q-3p) \leq \frac{3}{8}(2+p+q) \iff 3(p-q) \leq \frac{3}{8}(2+p+q) \iff$$

$$\iff 8(p-q) \leq 2+p+q \iff 0 \leq 9q+2-7p$$

Since $3p = 3(xy + yz + zx) \leq (x+y+z)^2 = 1$ and $9q \geq 4p-1$ (Schure Inequality $\sum_{cyc} x(x-y)(x-z) \geq 0$ normalized by $x+y+z=1$ in p-q notation) then

$$9q+2-7p = (9q-4p+1) + (1-3p) \geq 0$$

Arkady Alt

W33. (Solution by the proposer.)

1). Using the Young identity we get

$$\int_a^b f(x) dx + \int_{f(a)}^{f(b)} f^{-1}(x) dx = bf(b) - af(a)$$

or

$$\int_a^b f(x) dx + \int_{\lambda a}^{\lambda b} f^{-1}(x) dx = \lambda(b^2 - a^2) = \int_a^b 2\lambda x dx$$

or

$$\int_a^b f(x) dx + \lambda \int_a^b f^{-1}(\lambda x) dx = \int_a^b 2\lambda x dx$$

or

$$\int_a^b (f(x) + \lambda f^{-1}(\lambda x) - 2\lambda x) dx = 0$$

therefore exist $x_0 \in (a, b)$ such that

$$f(x_0) + \lambda f^{-1}(\lambda x_0) = 2\lambda x_0$$

2). Using the Young identity we get

$$\int_a^b f(x) dx + \int_{f(a)}^{f(b)} f^{-1}(x) dx = bf(b) - af(a)$$

or

$$\int_a^b f(x) dx + \int_{\lambda a}^{\lambda b} f^{-1}(x) dx = 0$$

or

$$\int_a^b f(x) dx - \lambda \int_a^b f^{-1}(\lambda x) dx = 0$$

or

$$\int_a^b (f(x) - \lambda f^{-1}(\lambda x)) dx = 0$$

therefore exist $x_0 \in (a, b)$ such that

$$f(x_0) - \lambda f^{-1}(\lambda x_0) = 0$$

or

$$f\left(\frac{f(x_0)}{\lambda}\right) = \lambda x_0$$

W34. (Solution by the proposer.) We have the followings:

$$\begin{aligned} \left(\frac{((k+2)!)^3}{2(k+1)(k+2)^2} \right)^{\frac{1}{k}} &= ((1 \cdot 2 \cdot 3)(2 \cdot 3 \cdot 4) \dots (k(k+1)(k+2)))^{\frac{1}{k}} \leq \\ &\leq \frac{1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \dots + k(k+1)(k+2)}{k} = \frac{(k+1)(k+2)(k+3)}{4} \end{aligned}$$

$$\begin{aligned} \sum_{k=1}^n \left(\frac{2(k+1)(k+2)^2}{((k+2)!)^3} \right)^{\frac{1}{k}} &\geq 4 \sum_{k=1}^n \frac{1}{(k+1)(k+2)(k+3)} = \\ &= 2 \sum_{k=1}^n \left(\frac{1}{(k+1)(k+2)} - \frac{1}{(k+2)(k+3)} \right) = \frac{n(n+5)}{3(n+2)(n+3)} \end{aligned}$$

Second solution. Noting that

$$\begin{aligned} \frac{((n+2)!)^3}{2(n+1)(n+2)^2} &= \frac{((n+2)!)^3}{2(n+1)(n+2)^2} = \\ &= \frac{((n+2)!)^3}{2} \cdot \frac{n!}{(n+1)!} \cdot \frac{((n+1)!)^2}{((n+2)!)^2} = \\ &= \frac{n! \cdot (n+1)! \cdot (n+2)!}{2} = \frac{1}{2} \prod_{k=1}^n k \cdot \prod_{k=1}^{n+1} k \cdot \prod_{k=1}^{n+2} k = \\ &= \frac{1}{2} \prod_{k=1}^n k \cdot \prod_{k=2}^{n+1} k \cdot 2 \prod_{k=3}^{n+2} k = \prod_{k=1}^n k \cdot \prod_{k=1}^n (k+1) \cdot \prod_{k=1}^n (k+2) = \\ &= \prod_{k=1}^n k(k+1)(k+2) \end{aligned}$$

by AM-GM Inequality we obtain

$$\begin{aligned} \left(\frac{((k+2)!)^3}{2(k+1)(k+2)^2} \right)^{\frac{1}{k}} &= \left(\prod_{i=1}^k i(i+1)(i+2) \right)^{\frac{1}{k}} \leq \\ &\leq \frac{1}{k} \sum_{i=1}^k i(i+1)(i+2) = \frac{1}{k} \cdot \frac{k(k+1)(k+2)(k+3)}{4} = \frac{(k+1)(k+2)(k+3)}{4}. \end{aligned}$$

Hereof,

$$\left(\frac{2(k+1)(k+2)^2}{((k+2)!)^3} \right)^{\frac{1}{k}} \geq \frac{4}{(k+1)(k+2)(k+3)}$$

and, therefore,

$$\begin{aligned}
& \sum_{k=1}^n \left(\frac{2(k+1)(k+2)^2}{((k+2)!)^3} \right)^{\frac{1}{k}} \geq \sum_{k=1}^n \frac{4}{(k+1)(k+2)(k+3)} \geq \\
& \geq \sum_{k=1}^n \left(\frac{2}{(k+1)(k+2)} - \frac{2}{(k+2)(k+3)} \right) = \frac{2}{(1+1)(1+2)} - \frac{2}{(n+2)(n+3)} = \\
& = \frac{1}{3} - \frac{2}{(n+2)(n+3)} = \frac{n(n+5)}{3(n+3)(n+2)}.
\end{aligned}$$

Arkady Alt

W35. (Solution by the proposer.) We have the followings:

$$1 \leq m^3 + m^2 - 1 < m^3 + m^3 + m^3 = 3m^3 < m^5$$

for all $m \geq 2$, and

$$1 \leq m^2 - m + 1 < m^2 + m^2 + m^2 = 3m^2 < m^4$$

Therefore $m^3 + m^2 - 1$ appear in product $(1 \cdot 2 \cdot \dots \cdot m)^5$ and $m^2 - m + 1$ appear in product $(1 \cdot 2 \cdot \dots \cdot m)^4$ therefore

$$m^5 + m - 1 = (m^3 + m^2 - 1)(m^2 - m + 1)$$

appear in product $(1 \cdot 2 \cdot \dots \cdot m)^9$. Therefore for $n = m^9$ we have infinitely many solutions.

W36. (Solution by the proposer.) First we will prove inequality

$$\triangle(a, b, c) \triangle(a^3, b^3, c^3) \leq 25F^4 = \triangle^2(a^2, b^2, c^2) \quad (1)$$

Due to homogeneity of inequality assume that semiperimeter s equal 1. Then, denoting $x := 1 - a$, $y := 1 - c$ and $p = xy + yz + zx$, $q := xyz$ we obtain $x + y + z = 1$, $x, y, z > 0$, $a = 1 - x$, $b = 1 - y$, $c = 1 - z$ and

$$\sum_{cyc} a^3 b^3 = \sum_{cyc} (x + yz)^3 = \sum_{cyc} (x^3 + 3x^2 yz + 3xy^2 z^2 + x^3 y^3) =$$

$$1 + 3q - 3p + 3q + 3pq + p^3 + 3q^2 - 3pq$$