U99. Let a and b be positive real numbers such that $a+b=a^4+b^4$. Prove that

$$a^a b^b \le 1 \le a^{a^3} b^{b^3}.$$

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First solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

If $x=1$ equality is clear, both sides of the inequality being zero.

If $x>1$, $\ln x = \int_1^x \frac{dz}{z} < \int_1^x dz = x-1$, since $z>1$ in the open integration interval.

If $x<1$, $\ln x = -\int_x^1 \frac{dz}{z} > -\int_x^1 dz = x-1$, since again $z>1$ in the open integration interval.

Taking $x=\frac{1}{a}$ easily produces $a^3 \ln a \ge a^3 - a^2$, while taking $x=a$ results in $a\ln a \le a^2 - a$, and similarly for b. Since the problem is equivalent to showing that $a\ln a + b\ln b \le 0 \le a^3 \ln a + b^3 \ln b$, it suffices to prove that, given positive reals a,b such that $a+b=a^4+b^4$, then $a^3+b^3 \ge a^2+b^2$ and $a^2+b^2 \le a+b$. The problem will be finished by proving these last two inequalities.

Define first $f(x)=a^x+b^x$. Clearly, $f'(x)=a^x\ln a + b^x \ln b$ and consequently $f''(x)=a^x\ln^2 a + b^x \ln^2 b \ge 0$, or f is convex, strictly unless $a=b=1$, and since $f(1)=f(4)$, then $f(2)\le f(1)$, yielding $a^2+b^2 \le a+b$, with equality iff $a=b=1$.

Note finally that, since $8(a+b)=8(a^4+b^4)\ge (a+b)^4$, where the inequality between arithmetic and quartic means has been used, then $ab \le \frac{(a+b)^2}{4} \le 1$ because of the AM-GM inequality, with equality iff $a=b=1$, and

$$(a+b+1)(a^3+b^3-a^2-b^2) = (a+b-a^2-b^2)(1-ab) \ge 0,$$

with equality iff $a=b=1$. The conclusion follows, and both proposed inequalities turn into equalities iff $a=b=1$.

Second solution by Arkady Alt, San Jose, California, USA

Due to the symmetry of constrain we can assume that $a \ge b$. Since

$a+b=a^4+b^4$ can be equivalently rewritten in the form

$a^3\left(1+\left(\frac{b}{a}\right)^4\right) = 1+\frac{b}{a}$ or in the form $b^3\left(1+\left(\frac{a}{b}\right)^4\right) = 1+\frac{a}{b}$ then,

denoting $t := \frac{a}{b}$, we obtain following parametrization for a and $b:$

$a=\sqrt[3]{\dfrac{t^4+t^3}{t^4+1}}\ ,\ b=\sqrt[3]{\dfrac{t+1}{t^4+1}}, t\geq 1.$

With this parametrization we have $a=bt$ and then:

1. $a^ab^b\leq 1 \iff \left(a^3\right)^a\left(b^3\right)^b\leq 1 \iff \left(a^3\right)^{bt}\left(b^3\right)^b\leq 1 \iff \left(a^3\right)^t b^3\leq 1 \iff$

$$\left(\frac{t^4+t^3}{t^4+1}\right)^t\frac{t+1}{t^4+1}\leq 1 \iff t^{3t}\leq\left(\frac{t^4+1}{t+1}\right)^{t+1} \iff t^3\leq\left(\frac{t^4+1}{t+1}\right)^{\frac{t+1}{t}},$$

where latter inequality is right, because applying Bernoulli's Inequality

(1) $\quad (1+x)^\alpha\geq 1+\alpha x,\ x>-1,\alpha\ \geq 1$

we obtain $\left(\dfrac{t^4+1}{t+1}\right)^{\frac{t+1}{t}}=\left(1+\dfrac{t^4-t}{t+1}\right)^{\frac{t+1}{t}}\geq 1+\dfrac{t^4-t}{t+1}\cdot\dfrac{t+1}{t}=t^3;$

2. $1\leq a^{a^3}b^{b^3} \iff 1\leq\left(a^3\right)^{a^3}\left(b^3\right)^{b^3} \iff 1\leq\left(a^3\right)^{t^3b^3}\left(b^3\right)^{b^3} \iff 1\leq\left(a^3\right)^{t^3}b^3 \iff$

$$1\leq\left(\frac{t^4+t^3}{t^4+1}\right)^{t^3}\frac{t+1}{t^4+1} \iff \left(\frac{t^4+t^3}{t^4+1}\right)^{t^3}\geq\frac{t^4+1}{t+1}=1+\frac{t\left(t^3-1\right)}{t+1}.$$

Consider two cases.

If $t\leq\sqrt[3]{2}$ then by Bernoulli's Inequality (1) we have

$\left(\dfrac{t^4+t^3}{t^4+1}\right)^{t^3}=\left(1+\dfrac{t^3-1}{t^4+1}\right)^{t^3}\geq 1+\dfrac{t^3\left(t^3-1\right)}{t^4+1}$ and

$\dfrac{t^3\left(t^3-1\right)}{t^4+1}\geq\dfrac{t^4-t}{t+1} \iff \dfrac{t^2}{t^4+1}\geq\dfrac{1}{t+1} \iff t^3+t^2\geq t^4+1 \iff$

$t^3(1-t)+\left(t^2-1\right)\geq 0 \iff (t-1)\left(t+1-t^3\right)\geq 0$ because

$t+1-t^3\geq t+1-2=t-1\geq 0;$

If $t>\sqrt[3]{2}$ then $t^3>2$ and applying inequality

(2) $(1+x)^\alpha\geq 1+\alpha x+\dfrac{\alpha(\alpha-1)}{2}x^2,\ x>-1,\alpha\ \geq 2$ (can by obtained by application (1) to $h'(x)$where $h(x):=(1+x)^\alpha-1-\alpha x-\dfrac{\alpha(\alpha-1)}{2}x^2$)

we get $\left(\dfrac{t^4+t^3}{t^4+1}\right)^{t^3}=\left(1+\dfrac{t^3-1}{t^4+1}\right)^{t^3}\geq 1+t^3\cdot\dfrac{t^3-1}{t^4+1}+$

$\dfrac{t^3\left(t^3-1\right)}{2}\cdot\left(\dfrac{t^3-1}{t^4+1}\right)^2$. Thus, suffices to prove that

$\dfrac{t^3\left(t^3-1\right)}{t^4+1}+\dfrac{t^3\left(t^3-1\right)^3}{2\left(t^4+1\right)^2}\geq\dfrac{t\left(t^3-1\right)}{t+1} \iff$

$$(3) \qquad \frac{t^2}{t^4+1} + \frac{t^2\left(t^3-1\right)^2}{2\left(t^4+1\right)^2} - \frac{1}{t+1} \geq 0.$$

Multiplying left hand side of inequality (3) by $2\left(t^4+1\right)^2(t+1)$ we obtain $2t^2\left(t^4+1\right)(t+1) + t^2\left(t^3-1\right)^2(t+1) - 2\left(t^4+1\right)^2 = t^9 - t^8 + 2t^7 - 2t^5 - 4t^4 + 3t^3 + 3t^2 - 2 = (t-1)\left(t^8 + 2t^6 + 2t^5 - 4t^3 - t^2 + 2t + 2\right) = (t-1)\left(\left(t^8 - t^2\right) + 2\left(t^6 - t^3\right) + 2\left(t^5 - t^3\right) + 2t + 2\right) \geq 0.$