Undergraduate problems

U85. Evaluate

$$\text{a)} \sum_{k=1}^{\infty} \frac{1}{1^3+2^3+\cdots+k^3} \qquad\qquad \text{b)} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{1^3+2^3+\cdots+k^3}$$

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First solution by Jose Hernandez Santiago, Oaxaca, Mexico

a. From

$$\frac{1}{k^2(k+1)^2} = \left(\frac{2}{k+1}-\frac{2}{k}\right)+\frac{1}{k^2}+\frac{1}{(k+1)^2}$$

and the fact that $\sum_{k=1}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k}\right)=-1$ and $\sum_{k=1}^{\infty}\frac{1}{k^2}$ we conclude that

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{1}{1^3+2^3+\ldots+k^3} &= 2^2\sum_{k=1}^{\infty}\frac{1}{k^2(k+1)^2}\\
&= 2^2\left\{2\sum_{k=1}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k}\right)+\sum_{k=1}^{\infty}\frac{1}{k^2}+\sum_{k=1}^{\infty}\frac{1}{(k+1)^2}\right\}\\
&= 2^2\left(2(-1)+\frac{\pi^2}{6}+\left(\frac{\pi^2}{6}-1\right)\right)\\
&= 2^2\left(\frac{\pi^2}{3}-3\right)\\
&= \frac{4(\pi^2-9)}{3}.
\end{aligned}$$

b. The series development for $\ln 2$ is crucial here. Indeed,

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{1^3+2^3+\ldots+k^3} &= 2^2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2(k+1)^2} \\
&= 2^2 \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{2}{k+1} - \frac{2}{k} + \frac{1}{k^2} + \frac{1}{(k+1)^2} \right) \\
&= 2^2 \cdot 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k+1} - 2^2 \cdot 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \\
&+ 2^2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} + 2^2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k+1)^2} \\
&= 2^2 \cdot 2(1-\ln 2) - 2^2 \cdot 2(\ln 2) \\
&+ 2^2 \left(\frac{\pi^2}{12} \right) + 2^2 \left(1 - \frac{\pi^2}{12} \right) \\
&= 4(3 - 4\ln 2).
\end{aligned}$$

Second solution by Arkady Alt, San Jose, California, USA

a) Since $1^3+2^3+...+k^3 = \dfrac{k^2\,(k+1)^2}{4}$ then

$$\begin{aligned}
\frac{1}{1^3+2^3+\cdots+k^3} &= \frac{4}{k^2\,(k+1)^2} \\
&= 4\left(\frac{1}{k} - \frac{1}{k+1}\right)^2 = 4\left(\frac{1}{k^2} + \frac{1}{(k+1)^2} - \frac{2}{k\,(k+1)}\right) \\
&= \frac{4}{k^2} + \frac{4}{(k+1)^2} - 8\left(\frac{1}{k} - \frac{1}{k+1}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{1^3+2^3+\cdots+k^3} &= 4\left(\sum_{k=1}^{n} \frac{1}{k^2} + \sum_{k=1}^{n} \frac{1}{(k+1)^2}\right) \\
&- 8\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = 8\sum_{k=1}^{n}\frac{1}{k^2} - 4 - \frac{4}{(n+1)^2} + 8\left(1 - \frac{1}{n+1}\right) \\
&= 8\sum_{k=1}^{n}\frac{1}{k^2} + 4 - \frac{4}{(n+1)^2} - \frac{8}{n+1}
\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{1^3+2^3+...+k^3} &= 4\sum_{k=1}^{n}\left(\frac{(-1)^{k-1}}{k^2}-\frac{(-1)^k}{(k+1)^2}\right)\\ &\quad -8\left(\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}+\sum_{k=1}^{n}\frac{(-1)^k}{k+1}\right)\\ &= 4-\frac{4(-1)^n}{(n+1)^2}-8\left(\sum_{k=1}^{n}\frac{2(-1)^{k-1}}{k}-1+\frac{(-1)^n}{n+1}\right)\\ &= 12-16\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}-\frac{4(-1)^n}{(n+1)^2}-\frac{8(-1)^n}{n+1}.\end{aligned}$$

We will use the following identities

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2}=\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6},$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}=\ln 2,$$

$$\lim_{n\to\infty}\left(\frac{4}{(n+1)^2}-\frac{8}{n+1}\right)=0,$$

$$\lim_{n\to\infty}\left(\frac{4(-1)^n}{(n+1)^2}+\frac{8(-1)^n}{n+1}\right)=0$$

to conclude that

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{1^3+2^3+\cdots+k^3} &= \lim_{n\to\infty}\left(8\sum_{k=1}^{n}\frac{1}{k^2}+4-\frac{4}{(n+1)^2}-\frac{8}{n+1}\right)\\ &= \frac{4\pi^2}{3}+4,\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{1^3+2^3+\cdots+k^3} &= \lim_{n\to\infty}\left(12-16\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}-\frac{4(-1)^n}{(n+1)^2}-\frac{8(-1)^n}{n+1}\right)\\ &= 12-16\ln 2\end{aligned}$$

and we are done.