U84. Let f be a three times differentiable function on an interval I, and let $a, b, c \in I$. Prove that there exists $\xi \in I$ such that

$$f\left(\frac{a+2b}{3}\right)+f\left(\frac{b+2c}{3}\right)+f\left(\frac{c+2a}{3}\right)-f\left(\frac{2a+b}{3}\right)-f\left(\frac{2b+c}{3}\right)-f\left(\frac{2c+a}{3}\right)=$$

$$=\frac{1}{27}(a-b)(b-c)(c-a)f'''(\xi).$$

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First solution by Arkady Alt, San Jose, California, USA

Let $g(t) := f\left(\frac{a+b+c}{3}+t\right)-f\left(\frac{a+b+c}{3}-t\right)$ and let $x=\frac{b-c}{3}$, $y=\frac{c-a}{3}$,

$z=\frac{a-b}{3}$ then $x+y+z=0$ and $\delta(a,b,c) := f\left(\frac{a+2b}{3}\right)+f\left(\frac{b+2c}{3}\right)+f\left(\frac{c+2a}{3}\right)-f\left(\frac{2a+b}{3}\right)-f\left(\frac{2b+c}{3}\right)-f\left(\frac{2c+a}{3}\right)=g(x)+g(y)+g(z)$.

We will consider non-trivial case where $x, y, z \neq 0$.

Note that if $I=(p,q)$ then $x, y, z \in (p_1, q_1)$ where $p_1 := p-\frac{a+b+c}{3}$ and $q_1 := q-\frac{a+b+c}{3}$ and g is three times differentiable function on the interval (p_1, q_1).

Since $g(0)=0$ and $g''(0)=0$ then by Maclaurin's Theorem

(1) $\quad g(t)=g'(0)\,t+\frac{g'''(\theta)\,t^3}{6}$ fore some $\theta \in (p_1, q_1)$.

Applying (1) to $t=x, y, z$ we obtain

$g(x)+g(y)+g(z)=\frac{g'''(\theta_x)\,x^3+g'''(\theta_z)\,y^3+g'''(\theta_z)\,z^3}{6}$ (because $x+y+z=0$).

Since $g'''(t) := f'''\left(\frac{a+b+c}{3}+t\right)+f'''\left(\frac{a+b+c}{3}-t\right)$ then

$\delta(a,b,c)=\frac{1}{6}\sum_{cyc} x^3\left(f'''\left(\frac{a+b+c}{3}+x\right)+f'''\left(\frac{a+b+c}{3}-x\right)\right)$ and by Darboux's Theorem about intermediate values of derivative for differentiable function f'' we there is such $\xi \in I$ such that

$$\sum_{cyc} x^3 \left(f''' \left(\frac{a+b+c}{3} + x \right) + f''' \left(\frac{a+b+c}{3} - x \right) \right) = 2 \left(x^3 + y^3 + z^3 \right) f''' (\xi) .$$

Thus $\delta (a, b, c) = \dfrac{\left(x^3 + y^3 + z^3 \right) f''' (\xi)}{3}$ and, because $x^3 + y^3 + z^3 = 3xyz,$ we finally

obtain $\delta (a, b, c) = \dfrac{1}{27} (a - b) (b - c) (c - a) f''' (\xi) .$

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Note first of all that we may choose wlog $c > b > a$, since exchanging any two of these values, inverts the sign of both sides of the given equation. Define now, for $m \neq 0$, and suitable parameters to be defined later $\Delta_1, \Delta_2 > 0$, functions $f_3(x), g_3(x)$:

$$f_3(x) = \frac{f(x + \Delta_1 + \Delta_2) - f(x - \Delta_1 + \Delta_2) - f(x + \Delta_1 - \Delta_2) + f(x - \Delta_1 - \Delta_2)}{4\Delta_1\Delta_2},$$

$$g_3(x) = m(x - x_3) + h_3.$$

Assume now that x_3 and $\Delta_3 > 0$ are chosen in a way such that

$$H_3 = (x_3 - \Delta_3, x_3 + \Delta_3) \subset I.$$

Obviously, $f_3(x)$ and $g_3(x)$ are differentiable in the interval H_3. Therefore, by Cauchy's generalization of the mean value theorem, $x_2 \in H_3$ exists such that

$$f_3'(x_2) = \frac{f_3(x_3 + \Delta_3) - f_3(x_3 - \Delta_3)}{g_3(x_3 + \Delta_3) - g_3(x_3 - \Delta_3)} g_3'(x_2) = \frac{f_3(x_3 + \Delta_3) - f_3(x_3 - \Delta_3)}{2\Delta_3}.$$

Using now this value of x_2, define functions $f_2(x), g_2(x)$:

$$f_2(x) = \frac{f'(x + \Delta_1) - f'(x - \Delta_1)}{2\Delta_1},$$

$$g_2(x) = m(x - x_2) + h_2.$$

Note that

$$f_3'(x) = \frac{f_2(x + \Delta_2) - f_2(x - \Delta_2)}{2\Delta_2}$$

Assume again that Δ_2 is chosen such that

$$H_2 = (x_2 - \Delta_2, x_2 + \Delta_2) \subset I.$$

Again, $f_2(x)$ and $g_2(x)$ are differentiable in H_2, and $x_1 \in H_2$ exists such that

$$f_2'(x_1) = \frac{f_2(x_2 + \Delta_2) - f_2(x_2 - \Delta_2)}{g_2(x_2 + \Delta_2) - g_2(x_2 - \Delta_2)} g_2'(x_1) = \frac{f_2(x_2 + \Delta_2) - f_2(x_2 - \Delta_2)}{2\Delta_2} =$$