Undergraduate problems

U79. Let $a_1 = 1$ and $a_n = a_{n-1} + \ln n$. Prove that the sequence $\sum_{i=1}^{n} \frac{1}{a_i}$ is divergent.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

First solution by Arin Chaudhuri, North Carolina State University, NC, USA

Note $a_n = a_1 + (a_2 - a_1) + \cdots + (a_n - a_{n-1}) = 1 + \ln(2) + \cdots + \ln(n) = 1 + \ln(n!)$.
From Stirling's approximation we have

$$\ln n! = \ln(\sqrt{2\pi}) - n + n\ln n + \frac{1}{2}\ln n + c_n$$

where $c_n \to 0$. Hence,

$$a_n = C - n + n\ln n + \frac{1}{2}\ln n + c_n$$

where $C = 1 + \ln(\sqrt{2\pi})$.
If $n \geq 2$ then diving throughout by $n \ln n$ we have

$$\frac{a_n}{n\ln n} = \frac{C}{n\ln n} - \frac{1}{\ln n} + 1 + \frac{1}{2n} + \frac{c_n}{n\ln n}$$

Note all terms above vanish as $n \to \infty$ except 1. Hence

$$\lim_{n\to\infty} \frac{a_n}{n\ln n} = 1 \tag{1}$$

Hence we can find an N such that for all $n \geq N$

$$\frac{a_n}{n\ln n} \leq 2$$

Hence for all $n \geq N$

$$\frac{1}{2n\ln n} \leq \frac{1}{a_n}$$

Using the well known result that $\sum_{k=2}^{\infty} \frac{1}{k\ln k} = +\infty$, we have $\sum_{k=1}^{\infty} \frac{1}{a_k} = +\infty$.

Second solution by Arkady Alt, San Jose ,California, USA

Since $n! < \left(\frac{n}{2}\right)^n$ and $a_n - a_1 = \sum\limits_{k=2}^{n} (a_k - a_{k-1}) = \sum\limits_{k=2}^{n} \ln k = \ln n!$ we get

$$a_n = 1 + \ln n!.$$

Note that $a_n < 1 + n\ln\left(\frac{n}{2}\right) < n\ln n, n \geq 2 \iff \frac{1}{a_n} > \frac{1}{n\ln n}, n \geq 2.$

Moreover, $\frac{1}{a_n} > \ln\ln(n+1) - \ln\ln n$, for $n \geq 2$ because by the Mean Value Theorem for some $c_n \in (n, n+1)$ we have

$$\ln\ln(n+1) - \ln\ln n = \frac{1}{c_n \ln c_n} < \frac{1}{n\ln n}.$$

Hence,

$$\sum_{k=1}^{n} \frac{1}{a_n} = 1 + \sum_{k=2}^{n} \frac{1}{a_n} > 1 + \sum_{i=2}^{n} \left(\ln\ln(k+1) - \ln\ln k\right) =$$

$$= 1 + \ln\ln(n+1) - \ln\ln 2,$$

and, therefore, sequence $\sum\limits_{k=1}^{n} \frac{1}{a_n}$ is divergent.

Third solution by Jean Mathieux, Senegal

We have that $a_3 \leq 3\ln 3$. Suppose that for $n > 3, a_n \leq n\ln n$, then

$$a_{n+1} \leq n\ln n + \ln(n+1) \leq (n+1)\ln(n+1).$$

So for all $i > 2$, $\frac{1}{a_i} \geq \frac{1}{i\ln 1}$. Also, since $t \to \frac{1}{t\ln t}$ is decreasing, $\int_i^{i+1} \frac{1}{t\ln t}dt \leq \frac{1}{i\ln i.}$ Thus

$$\sum_{i=3}^{n} \frac{1}{a_i} \geq \int_3^{n+1} \frac{1}{t\ln t}dt = \ln(\ln(n+1)) - \ln(\ln(3)).$$

Hence the given sequence is divergent.

Also solved by Magkos Athanasios, Kozani, Greece; Brian Bradie, Christopher Newport University, USA; Daniel Lasaosa, Universidad Publica de Navarra, Spain; John T. Robinson, Yorktown Heights, NY, USA; Perfetti Paolo, Dipartimento di matematica Universita degli studi di Tor Vergata, Italy; Vicente Vicario Garcia, Huelva, Spain; Roberto Bosch Cabrera, Faculty of Mathematics, University of Havana, Cuba.