U76. Let $f : [0,1] \to \mathbb{R}$ be an integrable function such that $\int_0^1 xf(x)dx = 0$. Prove that $\int_0^1 f^2(x)dx \geq 4\left(\int_0^1 f(x)dx\right)^2$.

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First solution by Arkady Alt, San Jose, California, USA

Since

$$\int_0^1 f(x)\,dx = \int_0^1 \left(1 - \frac{3x}{2} + \frac{3x}{2}\right) f(x)\,dx$$
$$= \int_0^1 \left(1 - \frac{3x}{2}\right) f(x)\,dx + \frac{3}{2}\int_0^1 xf(x)\,dx$$
$$= \int_0^1 \left(1 - \frac{3x}{2}\right) f(x)\,dx$$

and $\int_0^1 \left(1 - \frac{3x}{2}\right)^2 dx = \frac{1}{4}$. Then by the Cauchy Inequality

$\left(\int_0^1 f(x)\,dx\right)^2 = \left(\int_0^1 \left(1 - \frac{3x}{2}\right) f(x)\,dx\right)^2 \leq \int_0^1 \left(1 - \frac{3x}{2}\right)^2 dx \cdot \int_0^1 f^2(x)\,dx = \frac{1}{4}\int_0^1 f^2(x)\,dx.$

Equality occurs if $f(x) = 1 - \frac{3x}{2}$. Next we see that

$$\int_0^1 f(x)\,dx = \int_0^1 \left(1 - \frac{3x}{2}\right) dx = \frac{1}{4}, \quad \int_0^1 xf(x)\,dx = \int_0^1 x\left(1 - \frac{3x}{2}\right) dx = 0$$

and $\int_0^1 f^2(x)\,dx = \int_0^1 \left(1 - \frac{3x}{2}\right)^2 = \frac{1}{4}.$

Second solution by Li Zhou, Florida, USA