Undergraduate problems

U73. Prove that there is no polynomial $P \in \mathbb{R}[X]$ of degree $n \geq 1$ such that $P(x) \in \mathbb{Q}$ for all $x \in \mathbb{R} \backslash \mathbb{Q}$.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

First solution by Andrei Frimu, Chisinau, Moldova

If there were such q polynomial than we could build an injection $f : \mathbb{R} \setminus \mathbb{Q} \to \{0, 1, 2, \ldots, n\} \times \mathbb{Q}$ in the following way: take some $t \in \mathbb{R} \backslash \mathbb{Q}$. Let $P(t) = y \in \mathbb{Q}$. The equation $P(x) = y$ has $k \leq n$ solutions. Let them be $t_1 < t_2 < \ldots < t_k$. Clearly $t = t_i$ for some $1 \leq i \leq k \leq n$. Define $f(t) = (i, y)$. It is clear why this function is injective. The set $\{0, 1, 2, \ldots, n\} \times \mathbb{Q}$ is countable, hence $\mathrm{Im} f$ must be countable too. Then $g : \mathbb{R} \setminus \mathbb{Q} \to \mathrm{Im} f$, $g(x) = f(x)$ is a bijection, so g^{-1} exists, hence $\mathbb{R} \setminus \mathbb{Q}$ is countable, impossible.

Second solution by Arkady Alt, San Jose, California, USA

We will prove the statement of problem using induction on the degree $n \geq 1$.

Suppose that $P(x) = ax + b$, where $a, b \in \mathbb{R}$and $a \neq 0$, such that $P(x) \in \mathbb{Q}$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$. Since $x + 1, \dfrac{x}{2} \in \mathbb{R} \setminus \mathbb{Q}$ and $P(x+1), P\left(\dfrac{x}{2}\right) \in \mathbb{Q}$ then

$a = P(x+1) - P(x) \in \mathbb{Q}$and $b = 2P\left(\dfrac{x}{2}\right) - P(x) \in \mathbb{Q}$.

Hence, $x = \dfrac{P(x) - b}{a} \in \mathbb{Q}$ and that contradicts that $x \in \mathbb{R} \setminus \mathbb{Q}$.

Let $n \geq 2$. Suppose that the statement of problem holds for polynomials of degree

$m \in \{1, 2, ..., n-1\}$ we should to prove that there is no polynomial $P \in \mathbb{R}[X]$ of degree n such that $P(x) \in \mathbb{Q}$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$. Suppose the opposite

$P(x) = a_0 x^n + a_1 x^{n-1} + \cdot + a_{n-1} x + a_n$, where $a_0 \neq 0,$ holds $P(x) \in \mathbb{Q}$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$.

Since $x + 1 \in \mathbb{R} \setminus \mathbb{Q}$ then $P(x+1) \in \mathbb{Q}$ and for $P_1(x) := P(x+1) - P(x)$ holds

$1 \leq \deg P_1(x) < n,\ P_1(x) \in \mathbb{Q}$ for any $x \in \mathbb{R} \setminus \mathbb{Q}$. Thus we get a contradiction with earlier asumption of the induction, and so we are done.

Third solution by G.R.A.20 Math Problems Group, Roma, Italy